Problem 1.1 Fourier Transforms and Simple Filtering

Justify your answers for full credit.

a. 10pts What is the CTFT of the unit pulse \( y(t) = \begin{cases} 1 & t \in [-\frac{1}{2}, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases} \) ?

b. 10pts If \( b(t) \) is a periodic signal with period \( T \), and it is represented by the Fourier Series

\[
b(t) = \sum_{k=-\infty}^{\infty} B_k e^{j\frac{2\pi}{T}kt},
\]

then what is the Fourier Series representation of \( c(t) = b(t - 1) \)?
c. 15pts Consider discrete-time signals with period 2. Model these as 2-d vectors. Consider an LTI system that has impulse response \( h(0) = 1, h(1) = 2 \). Write this system as a matrix and give its eigenvectors and corresponding eigenvalues.

d. 10pts A discrete time LTI system has DTFT \( 1 - e^{-j\omega} \). What is its impulse response?
Problem 1.2 True/False. Do at least two of the following for full credit.

If the bold statement is true, give a proof for it. If the statement is false, show a counterexample or proof that it is false.

a. 20 pts Let $L$ be a system that acts on continuous time signals as follows: $[Lx](t) = x(t)\cos(t)$. Then, $L$ is L.T.I.
b. 20 pts Let $L$ be an LTI system that acts on signals that are defined on $Z_N$, the
discrete time interval $\{0, 1, 2, \ldots, N - 1\}$ viewed as positions along the
circumference of a circle. Delays and shifts on such signals are to be interpreted in a “wrap around” manner with $[D_\tau x](t) = x(t - \tau \mod N)$.
Let $x_\omega(t) = e^{j\omega t}$. Then for every real $\omega$ there exists a constant $\lambda_\omega$
so that $Lx_\omega = \lambda_\omega x_\omega$. 
c. 20 pts Let L be a linear system that acts on continuous time signals. Let \( x_\omega(t) = e^{j\omega t} \). There exists a complex valued function \( \lambda(\omega) \) so that for a particular subset of real \( \omega \in \Omega \), the system L has the property that \( [Lx_\omega](t) = \lambda(\omega)x_\omega(t) \). Then, for the class of signals that can be written \( y(t) = \sum_{i=1}^{N} \alpha_i x_{\omega_i}(t) \) (where the \( \omega_i \in \Omega \)), the system L is LTI.
Problem 1.3  AM Modulation System

\[
\begin{align*}
&\quad \text{LPF} \quad H_1(\omega) \quad x(t) \quad \text{Mixer 1} \quad y(t) \quad \text{Mixer 2} \quad z(t) \quad \text{LPF} \quad H_1(\omega) \quad \hat{x}(t)
\end{align*}
\]

In the above continuous time system, consider the LPF to be ideal and to perfectly pass through all frequencies less than 2.

\[
H_1(\omega) = \begin{cases} 
1 & \text{if } |\omega| < 2 \\
0 & \text{otherwise}
\end{cases}
\]

and \(\omega_0 = 10\) so that

\[
y(t) = x(t) \cos(\omega_0 t)
\]

and

\[
z(t) = y(t) \cos(\omega_0 (t + \phi))
\]

a. 15pts Suppose \(\phi = 0\) and \(x(t) = \sin(t)\). What are \(y(t), z(t), \hat{x}(t)\)?
b. 15pts Suppose $x(t) = \sin(t)$ but $\phi \neq 0$. What is $\dot{x}(t)$ as a function of $\phi$? Please plot the power of $\dot{x}(t)$ as a function of $\phi$. 
c. 10pts Suppose now that \( y(t) \) was corrupted by some potentially interfering signal and so the input to mixer 2 was now \( y'(t) = y(t) + \sin(\omega_n t) \) rather than just \( y(t) \). For what values of \( \omega_n \) would you see an undesirable component in \( \hat{x}(t) \)?