CS 70 Discrete Mathematics and Probability Theory
Summer 2014 James Cook — Midterm 2

Thursday July 31, 2014, 12:40pm-2:00pm.

Instructions:

- Do not turn over this page until the proctor tells you to.
- Don't write any answers on the backs of pages (we won't be scanning those). There is an extra page at the end in case you run out of space.
- The exam has 10 pages (the last two are mostly blank).

PRINT your student ID: ____

PRINT AND SIGN your name: __________________, __________________ ______________________

(last) (first) (signature)

PRINT your discussion section and GSI (the one you attend): ____

Name of the person to your left: ____

Name of the person to your right: ____

Name of someone in front of you: ____

Name of someone behind you: ____

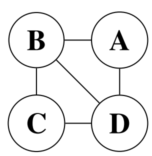
PRINT your name and student ID: ____

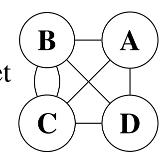
Short Answer

1. **(4 pts.) You flip a fair coin three times independently. Conditioned on the event that you get at least one heads (H), what's the probability that it came up heads all three times (HHH)?**

2. **(4 pts.) Suppose G is a connected undirected graph with n nodes. Two of the vertices have odd degree, and the rest have even degree. (So G does not have an Eulerian tour.)**

   Now, we add two uniformly random edges e_1 and e_2 to G. (We don't allow self-loops but we allow multiple edges, so there are $\binom{n}{2}^2$ equally likely ways to choose e_1 and e_2.)

   Example: if $n = 4$ and G is 

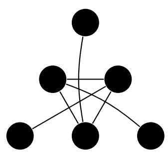
 and our random edges are (A,C) and (B,C), we get 

 .

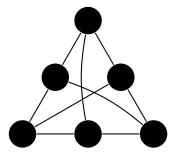
   What is the probablility that the new graph has an Eulerian tour? Your answer should be a function of n.

Short Answer (continued)

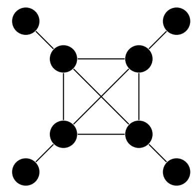
3. (4 pts.) Circle "Planar" below each planar graph and "Non-Planar" below each non-planar graph.



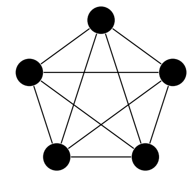
Planar / Non-Planar



Planar / Non-Planar



Planar / Non-Planar



Planar / Non-Planar

4. (6 pts.) Prove the following statement using a combinatorial proof.

$$\binom{2n}{3} = \binom{n}{3} + \binom{n}{3} + n\binom{n}{2} + \binom{n}{2}n$$

Counting

5. (12 pts.) Our new website, `cs70rules.com`, will require each student to choose a username satisfying the following two rules:

Rule 1. Usernames can only use upper-case letters A-Z. (So there are 26 possibilities for each character.)

Rule 2. Usernames must be six characters long.

For example, `STEVEN` is a valid username. `COOK85` and `PANDU` are not, because they violate Rules 1 and 2, respectively.

In the below questions, leave your answers as unevaluated expressions like "$\binom{20}{6} \times 7!$". You do not need to explain your answers.

(a) (2 pts.) How many possible usernames satisfy Rules 1 and 2?

(b) (2 pts.) Rule 3 is added: a username may not use a letter more than once. For example, `STEVEN` is no longer allowed, since it has two `E`s, but `JAMESC` is still allowed.

How many possible usernames are there that satisfy Rules 1-3?

Continued on the next page!

Counting (continued)

(c) (4 pts.) Rule 4 is added: the letters in a username must appear in alphabetical order. So `JAMESC` is not valid any more, but putting the letters in order, `ACEJMS` is valid. Similarly, `FEDCBA` is not valid but `ABCDEF` is valid.

How many possible usernames satisfy Rules 1-4?

(d) (4 pts.) After students start complaining, we remove Rules 3 and 4. We also relax Rule 1 so that a username can have up to one underscore (_).

For example, `AJAY_T` and `STEVEN` are allowed, but `I_AM_J` is not allowed, because it has more than one underscore.

How many possible usernames are there now?

PRINT your name and student ID: ____________________

Probability

6. (12 pts.) Pat has a deck with five cards, numbered $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$. She proposes the following game to Gary:

1. Gary takes two cards uniformly at random: call them A and B. Gary sees the number on A, but B is hidden so he can't see it.
2. Next, Pat takes a card C uniformly at random from the 3 that remain. (Gary doesn't see this either.)
3. Finally, Gary must choose whether to keep card A or B.

Whoever has the higher card wins.

Here's an example game. In Step 1, Gary takes $A = \boxed{3}$ and $B = \boxed{5}$. In Step 2, Pat takes $C = \boxed{4}$. Gary only knows that card A is $\boxed{3}$, and decides to keep A. Gary loses because $\boxed{4}$ is higher than $\boxed{3}$.

(a) (2 pts.) Gary models the game with a probability space. An outcome consists of the values of cards A, B and C. For example, one possible outcome is $(3,5,4)$, meaning Gary receives $A = \boxed{3}$ and $B = \boxed{5}$ and Pat recieves $C = \boxed{4}$.
If the probability distribution is uniform, then what is the probability of each outcome?

(b) (4 pts.) If Gary's strategy is to always keep card A, what's the probability that Gary wins?

Continued on the next page!

Probability (continued)

(c) (6 pts.) Gary tries a new strategy: if card A **is** $\boxed{5}$**, then he keeps card** A**, and otherwise keeps card** B**. What is the probability that Gary wins with this new strategy?**

PRINT your name and student ID: ________________________________

Conditional Probability

7. (9 pts.) Suppose there are two events, A and B. You are given the following information:
$\Pr[A|B] = \frac{1}{2}$, $\Pr[B|A] = \frac{1}{3}$, $\Pr[A] = \frac{2}{3}$.

(a) (3 pts.) What is $\Pr[A \cap B]$?

(b) (4 pts.) What is $\Pr[B]$?

(c) (2 pts.) Are A and B independent events?

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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[Doodle page! Draw us something if you want or give us suggestions or complaints.]