CS 70 Discrete Mathematics and Probability Theory

Summer 2014 James Cook

Final Exam

Friday August 15, 2014, 5:10pm-8:10pm.

Instructions:

- Do not turn over this page until the proctor tells you to.
- Don't write any answers on the backs of pages (we won't be scanning those). There is an extra page at the end in case you run out of space.
- The exam has 18 pages (the last two are mostly blank).

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________, ________________ ________________
(last) (first) (signature)

PRINT your discussion section and GSI (the one you attend): ________________________________

Name of the person to your left: ________________________________

Name of the person to your right: ________________________________

Name of someone in front of you: ________________________________

Name of someone behind you: ________________________________

Important note:

In any multi-part problem, you can refer to numbers from the previous parts, even if you couldn't solve them! Use A for the answer to part (a), B for the answer to part (b), etc.
For example, if you know that the answer to (b) should be 5 times the answer to (a), write $5A$ as your answer to (b).

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Important Note

Have you read the important note on the front of the exam?

True/False

1. [Law of the excluded middle](10 pts.) **For each of the following statements, circle T if it is true and F otherwise. You do not need to justify or explain your answers.**

T F $(\forall n \in \mathbf{N})\left((2n)! = \binom{2n}{n}(n!)^2\right)$

T F **$\mathbf{Q}$ and $\mathbf{N}^3$ have the same cardinality.** (Note: $\mathbf{N}^3$ is the set of triples of natural numbers, e.g. $(5,0,17) \in \mathbf{N}^3$.)

T F **If you throw n^2 balls into n^3 bins uniformly at random, the probability that at least one bin will have more than one ball will be less than 0.1 (for large enough n).**

Consider the following instance of the stable marriage problem.

Woman	Preferences		
A	1	2	3
B	1	2	3
C	3	1	2

Man	Preferences		
1	B	A	C
2	C	B	A
3	B	A	C

T F **In the above stable marriage instance, (1, B), (2, A), (3, C) is a stable pairing.**

T F **In the above stable marriage instance, 1 is A's optimal man.**

Short Answer

2. [U mod?](4 pts.) **Find an integer x such that $8x \equiv 20 \pmod{22}$.**
(Warning: $\gcd(22,8) \neq 1$.)

3. [Come Together](4 pts.) **The code to open your secret club's treasure chest is 15. Your club has 4 other members, named John, Paul, George and Ringo. What information can you give each of them so that any 3 of them can discover that the secret is 15, but if only 2 of them share their information, they cannot discover the secret? Fill in the boxes below.**

(If you need any random numbers, feel free to just use 0 if it makes your calculations easier.)

Scratch work:

Answer:

Tell all of them: "$P(x)$ is a polynomial of degree at most [ ]. The secret is $P($[ ]$)$."

Tell John: $P($[ ]$) \equiv$ [ ] (mod [ ]).

Tell Paul: [ ]

Tell George: [ ]

Tell Ringo: [ ]

Short Answer (continued)

4. [Connect the dots](3 pts.) **Find a polynomial $p(x)$ such that $p(1) \equiv 0 \pmod 7$, $p(4) \equiv 0 \pmod 7$ and $p(5) \equiv 1 \pmod 7$. You do not need to simplify your answer.**

(Hint: This shouldn't take very long.)

5. [What do you expect?](4 pts.) **Suppose X is a random variable with the following distribution:** $\Pr[X = 1] = \frac{1}{12}$, $\Pr[X = 2] = \frac{1}{6}$, $\Pr[X = 3] = \frac{3}{4}$. **What is the expected value of $\frac{1}{X^2}$?**

Short Answer (continued)

6. [He is a Cook after all...](4 pts.) **James is baking cakes. Every cake he bakes is rated one of E, O or T (for Excellent, just Okay, or Terrible). We record the sequence of ratings his cakes get.**

The cakes are rated randomly: E with probability $\frac{1}{2}$, O with probability $\frac{1}{4}$ and T with probability $\frac{1}{4}$. The ratings are independent. (For example, if he bakes 7 cakes, one possible sequence is EOETOEE, which occurs with probability $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} = 2^{-10}$.)

If James bakes 20 cakes, what is the probability that he gets 7 Es, 4 Os and 9 Ts?

7. [Dicey stuff](3 pts.) **Suppose you roll a standard fair die 100 times. Let X be the sum of the numbers that appear over the 100 rolls. Give a bound on the probability that X is between 300 and 400.**

Short Answer (continued)

8. [Probably Fun?](4 pts.) **Show that if A and B are events, then $\Pr[A \cap B] \geq \Pr[A] + \Pr[B] - 1$.**

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You won't believe what these numbers sum to!

9. (8 pts.) **Prove that for every** $n \in \mathbf{N}$**,**

$$\sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}.$$

Variation on a Theme

10. (10 pts.) **Ziva has a set of 10 magical dice named $D_1, D_2, \ldots, D_{10}$. When she rolls all 10 dice, the number 1 always appears at least once. Notice that there are $6^{10} - 5^{10}$ possible outcomes when she rolls the dice. Every one of these outcomes appears with probability $\frac{1}{6^{10} - 5^{10}}$ (so the dice follow the uniform distribution).**

Leave your answers below as unevaluated expressions like $\frac{5!}{6^{100}}$.

(a) (2 pts.) What is the probability that $D_1 = 1$? (The answer is not $\frac{1}{6}$.)

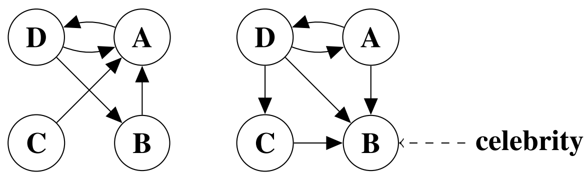
(b) (2 pts.) Let X be the number of dice that show 1. What is $\mathrm{E}(X)$?

(c) (6 pts.) What is $\mathrm{Var}(X)$?

Party at Ajay's House

11. (23 pts.) **Ajay's thinking of having a party at his house. He has invited n guests ($n \geq 2$). To keep track of which guests know which other guests, he's drawn a directed graph G with n nodes (one per guest), with an edge from guest x to guest y whenever x knows y.**

A guest x is called a *celebrity* if every other guest knows x, but x does not know any other guest. For example, below are two parties. The party on the left has no celebrities. In the other party, B is a celebrity.



(a) (4 pts.) If the party has a celebrity, can G have an Eulerian cycle (one that visits every edge once)? Can it have a Hamiltonian cycle (one that visits every node once)? Justify your answers.

For the remaining questions, suppose that G is generated randomly as follows. For every two guests x and y, x knows y with probability p, and all such events are independent. (Examples: The probability that x knows y and y also knows x is p^2. The probability that w knows x and y but doesn't know z is $p^2(1-p)$.)

(b) (4 pts.) What is the probability that every guest knows every other guest?

Continued on the next page!

Party at Ajay's House (continued)

(c) (3 pts.) Let's pick a particular guest u. What's the probability that u is a celebrity?

(d) (4 pts.) Let's pick a second guest v. What's the probability that u knows v given that u is not a celebrity?

(e) (4 pts.) What's the probability that one of u and v is a celebrity? *(Hint: a party cannot have more than one celebrity.)*

(f) (4 pts.) What's the probability that there are no celebrities?

Your Bassic Geometric Distribution

12. (9 pts.) **Hao has a computer with two speakers. Every day, each speaker fails with probability** p **independently. Let** L **and** R **be the number of days the left and right speakers last until failing. So** $L \sim \textbf{Geom}(p)$**,** $R \sim \textbf{Geom}(p)$**, and** L **and** R **are independent random variables.** (For example, if both speakers fail on the first day, then $L = R = 1$.)

In the below questions, simplify your answers as much as possible.

(a) (3 pts.) For an integer $k \geq 1$**, what is** $\Pr[L \geq k]$**?** (In other words, the probability that the left speaker fails on day k or later.)

(b) (3 pts.) Let X **be the number of days until both speakers have failed: in other words,** X **is the larger of** L **and** R**.**

For an integer $k \geq 1$**, what is** $\Pr[X < k]$**?**

(c) (3 pts.) For an integer $k \geq 1$**, what is** $\Pr[X = k]$**?**

Binomial Process

13. (16 pts.) **Three wise men, named W_1, W_2 and W_3, are flipping coins independently at each timestep until the end of time. Specifically, W_1 flips a coin with bias p_1 at every timestep, W_2 flips his coin with bias p_2 every timestep, W_3 flips his coin with bias p_3 every timestep, and all coin flips are independent.** (The *bias* of a coin is the probability that it comes up heads when you flip it.) **You are a lonely wanderer who comes upon the three wise men and starts observing them.**

Here is an example outcome:

Timestep	1	2	3	4	5	6	⋯
Wise Man W_1	T	H	T	T	T	T	⋯
Wise Man W_2	T	T	T	T	T	T	⋯
Wise Man W_3	H	H	T	H	T	T	⋯

(a) **(2 pts) Let X be the number of timesteps it takes for Wise Man W_1 to flip his first heads.** (In the example outcome above, $X = 2$.) **Give the probability distribution of X, along with any parameters necessary to describe it.**

(b) **(3 pts.) Let B be the probability that, on a given timestep, at least one of the coins comes up heads. Find B.**

Continued on the next page!

Binomial Process (continued)

(c) (2 pts.) If you look at the first 20 timesteps, let Y be the number of those timesteps where only tails were flipped. Give the probability distribution of Y, along with any parameters necessary to describe it.

(d) (3 pts.) On a particular timestep, you notice one wise man flipped heads, and the other two flipped tails. Given that, what's the probability that Wise Man W_1 was the one who flipped heads?

Continued on the next page!

Binomial Process (continued)

(e) (3 pts.) Let Z be the number of timesteps it takes to see 2 timesteps in a row with no heads (including those two time steps). (In the example outcome, $Z = 6$.) **Find** $\mathrm{E}(Z)$**.**

(f) (3 pts.) Let R be the event that the first time heads is flipped, it was Wise Man W_1 that flipped heads, and neither of the other wise men flip heads on that time step. (In the example outcome, R does not happen, since Wise Man W_3 flips heads first.) **Find** $\Pr[R]$**.**

Note: This problem corresponds to a very real thing called a Binomial process, which you just derived several of the key properties of! The all-too-important continuous analog is called a Poisson process (see EE 126), which is used to handle queues, hits on a website, traffic, and many other real-life problems!

[Extra page 1. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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[Extra page 3. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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[Doodle page! Draw us something if you want or give us suggestions or complaints.]