CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand Midterm 1

PRINT Your Name: ______________________________, ______________________________
(last) (first)

SIGN Your Name: ____________________________________

PRINT Your Student ID: ____________________________________

CIRCLE your exam room:
1 Pimentel 141 Mccone 159 Mulford 10 Evans 2040 VLSB 105 North Gate Other

Name of the person sitting to your left: ____________________________________

Name of the person sitting to your right: ____________________________________

- After the exam starts, please write your student ID (or name) on every odd page (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.
- On the short answer questions. You need only give the answer in the format requested (e.g., true/false, an expression, a statement.) We note that an expression may simply be a number or an expression with a relevant variable in it. **For short answer questions, correct clearly identified answers will receive full credit with no justification. Incorrect answers may receive partial credit.**
- On question 4-7, do give arguments, proofs or clear descriptions as requested, though there are one or two short answers there.
- You may consult one single-sided sheet of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, and computers are not permitted.
- There are 14 single sided pages on the exam. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and facts that were proven in the notes and/or in lecture.**
- **You have 120 minutes: there are 33 parts on this exam.**
  - **Problems 1-3: 19 parts. 50 points.**
  - **Problem 4-7: 14 parts. 50 points.**

Do not turn this page until your instructor tells you to do so.

1. A Modest Proposition: 3/3/3/3/3 Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

You have $Z = \forall x, (P(x) \implies (Q(x) \vee R(x)))$. State in each case below whether Z is certainly true, certainly false, or possibly true.

1. There is an x, such that $P(x)$ is true and either $Q(x)$ or $R(x)$ is true.

2. For every x, if $R(x)$ is false then $P(x)$ is false.

3. For every x where $(\neg Q(x) \wedge \neg R(x))$ is true, we have $P(x)$ is false.

4. For every x such that $Q(x)$ is false, then either $R(x)$ is true or $P(x)$ is false.

5. There is an x such that $Q(x)$ is false and $R(x)$ is false and $P(x)$ is true.

2. Short Answer/True/False/Maybe: 3/2/2/2/3/2/2/2/2 Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

1. Give a set of preferences, an unstable pairing for those preferences, and a rogue couple with respect to your pairing. (Again, we are asking for an *unstable* pairing.)

2. (True/False.) If the preference lists of the men are all the same, the men's least favorite woman is always paired with her least favorite man in the stable pairing returned by the traditional marriage algorithm.

3. (True/False.) Any pairing where more than one man is matched to his least favorite partner is unstable.

4. (True/False.) In a run of the TMA (the traditional marriage algorithm), on one day, if a woman accidentally rejects a man she prefers to a man she keeps on the string, then the algorithm must terminate with a rogue couple.

For the following questions. Consider a stable marriage instance on n men and n women: say one forms a graph consisting of vertices for each man and woman, and an edge for the *first* preference of each person. That is, if woman 1 prefers man A the most, there would be an edge $(1,A)$ in the resulting graph. Notice the graph may not be simple.

5. What is the maximum degree of the graph? (Short answer: an expression possibly involving n.)

6. (True/False) If the maximum degree of the graph is two, the female optimal pairing pairs every female with her favorite partner (favorite means first on her preference list).

7. (True/False) If the maximum degree of the graph is two, there are only two stable pairings.

This ends the question on the first preference graph for stable marriage.

8. (True/False) If $a|b$ and $b|c$ then $a|c$. (Recall that $x|y$ means that x divides y.)

9. How many edges need to be removed from a 3-dimensional hypercube to get a tree? (Short answer: a number.)

3. More Short Answer: 3/3/3/3/3 Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

1. If $13x = 5 \pmod{46}$, what is x? (Short answer.)

2. What is the maximum number of solutions for x in the range $\{0, \ldots, N-1\}$ for any equation of the form $ax = b \pmod{N}$, when $gcd(a,N) = d$? (Short answer: an expression possibly involving N, a, b, and/or d.)

3. What is $2^{50} \pmod{65}$? (Short answer: a number between 0 and 64 inclusive.)

4. What is the size of the range of the function $f(x) = px \pmod{pq}$, where the domain is $\{1, \ldots, pq-1\}$? The range of a function is the set of values y, where $f(x) = y$ where x is in the domain. (Note: the set $\{0 \mod 2, 1 \mod 2, 2 \mod 2\}$ has size 2, since $0 = 2 \mod 2$.) (Short answer: an expression possibly using p and/or q.)

5. What is the smallest number of colors that can be used to properly color a tree? Recall that a proper coloring is an assignment of colors to vertices where for each edge (u, v), u and v are assigned different colors. (Short answer.)

4. Some Proofs:3/6

1. Prove that for $x, y \in Z$, that if $x - y > 536$, then $x > 268$ or $y < -268$.

2. Show by induction that $\sum_{i=1}^{n} \frac{1}{i^3} \leq 2$.

5. Unique factorization. 4/3/4

In class, we proved that any number can be written as a product of primes. In this problem, you will prove that every number has a unique prime factorization. **(Warning: do not use the fact that the factorization is unique in this problem as the point is to prove this fact.)**

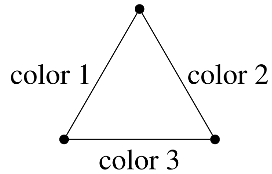
1. Prove that for a prime p that if $p|ab$ then $p|a$ or $p|b$. (You may use the fact that if $gcd(x,y) = 1$ that there are integers m and n where $mx + ny = 1$.)

2. Prove that if p is prime and $p|p_1 \cdot p_2 \cdots p_k$ that there is some i where $p|p_i$. (You may use part 1)

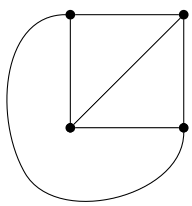
3. Prove that any natural number, $x \geq 2$ has a unique prime factorization; there is a unique multiset of primes whose product is x. For example, 12 is the product of the multiset $\{2,2,3\}$. And the multiset $\{2,3,2\}$ is the same multiset as $\{2,2,3\}$ but different from the multiset $\{2,3\}$. Perhaps view them as sorted. (You may use the result from part 2.)

6. Edge Colorings: 3/3/3/3/3/4

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors.



1. (Short Answer) Show that the 4 vertex complete graph below can be 3 edge colored (use the numbers $1, 2, 3$ for colors.)



2. (Short Answer) How many colors are required to edge color a 3 dimensional hypercube?

3. Prove that the complete graph on n vertices, K_n, can always be edge colored with n colors. (Hint: is $x+1 \pmod{n}$ a bijection?)

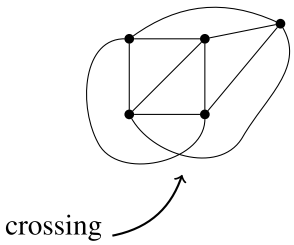
4. Prove that any graph with maximum degree d can be edge colored with $2d-1$ colors.

5. Show that any tree has a degree 1 vertex. (You may use any definition of a tree that we provided in the notes, homeworks or lectures to prove this fact.)

6. Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

7. Planar Graphs:3/4/4

K_5 can be drawn in the plane with exactly one crossing as follows.



1. Draw $K_{3,3}$, the complete bipartite graph with three vertices on each side, in the plane where there is exactly one crosssing.

2. Prove that K_6 cannot have a drawing in the plane with at most one crossing. (You may use the fact that for any planar graph with e edges and v vertices that $e \leq 3v - 6$.)

3. Prove that for any planar graph where every cycle has length at least 6, there is a vertex of degree at most 2. (You may use Euler's formula: that $v + f = e + 2$ for any planar drawing with f faces of a graph with e edges and v vertices.)