Midterm 2 Solutions

Note: These solutions are not necessarily model answers. Rather, they are designed to be tutorial in nature, and sometimes contain more explanation (occasionally much more) than an ideal solution. Also, bear in mind that there may be more than one correct solution. Following some of the solutions, there is some additional text in italics that explains some common mistakes. The maximum total number of points available is 60.

1. Coloring Hypercubes

(a) Let red and blue denote our colors. We will prove by induction on n that the n-dimensional hypercube is 2-vertex-colorable for every n. *5pts*

Base case: For $n = 1$, the hypercube is a single edge. If we color one vertex red and the other blue we have a 2-vertex coloring, since the adjacent vertices are colored differently.

Inductive Step: Assume we've shown this to hold for n-dimensional hypercubes, we will show this holds for $n + 1$-dimensional hypercubes. Recall that we can define an $n + 1$ dimensional hypercube as two n-dimensional hypercubes where every vertex i in the first hypercube is connected to vertex i in the second hypercube. Considering this definition, for a given $n + 1$ dimensional hypercube, let H_0, H_1 denote the first and second n-dimensional hypercubes, respectively.

By the inductive hypothesis, we assume that H_0 is 2-vertex colorable, and therefore there exists some coloring scheme which is a legal 2-vertex coloring of H_0. Given this coloring, we will color H_1 in the opposite coloring scheme which, given a color of vertex i in H_0 assigns the opposite color to the vertex i in H_1. Since we colored the vertices in both H_0 and H_1, we have colored all the vertices in the hypercube. It remains to show that this coloring scheme is legal.

Assume, for purpose of contradiction that there is a given vertex i in H_1 which has an adjacent neighbor colored with the same color. If that neighbor is in H_1, then this means that the coloring of H_1 is not legal. Observe that if a coloring scheme is a legal 2-vertex coloring on some graph G, then the opposite coloring scheme is also a legal 2-vertex coloring on G. Since we colored H_1 with the opposite scheme of H_0, and H_0 is identical to H_1, this implies that the coloring of H_1 is a legal 2-vertex coloring, which contradicts having two adjacent vertices in H_1 sharing the same color. If the neighbor is in H_0, then we know, by definition of the $n + 1$-dimensional hypercube, that the neighbor must be i in H_0. In our coloring however, we colored i in H_0 and i in H_1 in opposite colors, which again contradicts our assumption. Similarly, we can show for the case where i is in H_0.

The majority did well on this problem. A different approach that could have been taken is to color the vertices in the hypercube according to their parity. Some people gave an algorithm for coloring the vertices, though did not prove its correctness.

(b) We will again prove by induction on n, and show that the n-dimensional hypercube is n-edge-colorable for every n. *5pts*

Base case: For $n = 1$, the hypercube is a single edge, and using any color is a legal coloring.

Inductive Step: Assume we've shown this to hold for n-dimensional hypercubes, we will show this holds for $n+1$-dimensional hypercubes. We will again use the recursive definition of the hypercube as we did above, again denoting H_0 and H_1 as the two n-dimensional hypercubes that compose our $n+1$-dimensional hypercube. Observe that this definition partitions the set of edges to three disjoint sets: edges that are only in H_0, edges that are only in H_1 and edges which connect H_0 and H_1. According to our inductive hypothesis we know we can color the edges of H_0 and H_1 using n colors, so that no two adjacent edges have the same color. We can therefore color the edges of H_0 and H_1 according to these coloring schemes, and we will color the edges which connect between H_0 and H_1 with some different color, which is not used in the edge coloring of either H_0 or H_1. We have therefore colored all the edges of the hypercube with at most $n+1$ colors, and it remains to show that this coloring scheme is legal.

Assume for purpose of contradiction that there is a given edge e which has an identical color to one of its adjacent edges. If e is an edge in H_1, since we colored H_1 in a legal edge coloring, we know all its adjacent edges in H_1 are colored with a different color. Therefore it must be the case that the adjacent edge is an edge which connects H_1 and H_0. This however, contradicts our specification of using a new color for the connecting edges. A similar statement can be shown for an edge in H_0. If we assume that e is an edge that connects between H_1 and H_0, the previous argument shows that the adjacent edge that has the same color cannot be in H_1 and H_0. Therefore, the only possibility would be that there is some other edge that connects between H_1 and H_0 which is adjacent to e. Let i be the index for which e connects between i in H_0 and i in H_1. If there is an adjacent edge e' to e which connects between H_1 and H_0, it must be the case that e' connects between i in H_0 and some other vertex $j \neq i$ in H_1 or between $j \neq i$ in H_0 and i in H_1. This, however, contradicts our recursive definition of the $n+1$-dimensional hypercube.

A common error here was showing that we need **at least** *n colors to have a legal coloring of the n-dimensional hypercube, which is almost trivial. You should have shown that you need* **at most** *n colors for a legal coloring of the hypercube. Some people argued that each vertex has only n adjacent edges and therefore we can color them in different colors. To make this argument work one needs to show how all edges in the hypercube can be colored in a manner that guarantees that no two adjacent edges will be colored in the same color. Many people who used an approach similar to the one presented above did not prove why one can color the connecting edges in a new color and still have a legal coloring.*

2. Random Graphs

(a) Vertex 1 is isolated if and only if all of the possible edges connecting 1 to the other vertices in the graph are absent. The number of such edges is $n-1$, and each of them is absent independently with probability $1-p$. Hence we get *2pts*

$$\Pr[\text{1 is isolated}] = (1-p)^{n-1}.$$

Most people got this right. One fairly common minor mistake was to write $(1-p)^n$ instead of $(1-p)^{n-1}$, forgetting that there are only $n-1$ (rather than n) possible edges incident at a vertex. A more serious mistake was to write p^{n-1} instead of $(1-p)^{n-1}$, forgetting that the probability of an edge being absent is $1-p$, not p. Quite a few students wrote down a formula that bore little resemblance to any of the above, and seemed to misunderstand the question.

(b) Vertices 1 and 2 are both isolated if and only if the edge between them, and all edges connecting either of them to other vertices in the graph, are absent. The total number of such edges is $1+2(n-2) = 2n-3$. Hence, as in part (a), the probability is *3pts*

$$\Pr[\text{1 and 2 are isolated}] = (1-p)^{2n-3}.$$

An alternative approach is to use conditional probability. Let E_1, E_2 denote the events that vertex 1 is isolated and vertex 2 is isolated respectively. We want to compute $\Pr[E_1 \cap E_2]$. By the chain rule, this is given by $\Pr[E_1 \cap E_2] = \Pr[E_1] \times \Pr[E_2 \mid E_1]$. From part (a), we know that $\Pr[E_1] = (1-p)^{n-1}$. And it is easy to see that $\Pr[E_2 \mid E_1] = (1-p)^{n-2}$, because once we know that vertex 1 is isolated we know that edge $\{1,2\}$ is absent, so we only need to rule out $n-2$ additional edges to ensure that vertex 2 is isolated. Hence $\Pr[E_1 \cap E_2] = (1-p)^{n-1} \times (1-p)^{n-2} = (1-p)^{2n-3}$.

A common incorrect answer here was $(1-p)^{2n-2}$, which is obtained by assuming that the events that vertex 1 and vertex 2 are isolated are independent and therefore just squaring the probability from part (a), or equivalently, by forgetting that the edge $\{1,2\}$ is incident on both vertices and double-counting it when excluding edges.

(c) Let E_1, E_2 be the events that vertex 1 is isolated and vertex 2 is isolated respectively. Our goal is to compute $\Pr[E_1 \cup E_2]$. By inclusion-exclusion, we have *3pts*

$$\Pr[E_1 \cup E_2] = \Pr[E_1] + \Pr[E_2] - \Pr[E_1 \cap E_2].$$

But from part (a) we have $\Pr[E_1] = \Pr[E_2] = (1-p)^{n-1}$, and from part (b) we have $\Pr[E_1 \cap E_2] = (1-p)^{2n-3}$. Hence the desired probability is

$$2(1-p)^{n-1} - (1-p)^{2n-3}.$$

A common error here was to incorrectly quote the inclusion-exclusion principle by changing the minus sign to a plus sign. Another error was to simply sum $\Pr[E_1] + \Pr[E_2]$, which is incorrect because E_1 and E_2 are not disjoint events. Finally, a surprising number of students seemed to think that $\Pr[E_1]$ and $\Pr[E_2]$ are different!

(d) In order that vertices 1,2,3 form an isolated triangle, we need the three edges $\{1,2\}$, $\{2,3\}$ and $\{1,3\}$ to be present, and all other edges between these three vertices and the rest of the graph to be absent; the number of such other edges is $3(n-3)$. Hence the probability we want is *3pts*

$$p^3(1-p)^{3(n-3)}.$$

Some people got very confused in this part. Among those who got it almost (but not quite) right, the most common error was to forget one of the factors of $(n-3)$ in the exponent of $(1-p)$. This usually resulted from trying to use conditional probability and the chain rule, but not being careful enough about computing the conditional probabilities.

3. Counting

(a) In this problem, we are allowed to encode letters by any string of dots and dashes, which has length *at most* 10. Thus, we count separately the number of letters that can be formed using *exactly* i dots and dashes for all $1 \le i \le 10$. There are 2^i different strings of length i that can be formed using dots and dashes (since for every position of the string, we have a choice of two symbols). Hence, the total number of letters that can be formed are *3pts*

$$\sum_{i=1}^{10} 2^i \; = \; 2046$$

The most common error for this part was to ignore the "at most" and consider only strings of length exactly 10.

(b) For each pair of vertices $\{u,v\}$ in the graph, we are deciding whether to put an edge between them or not. Hence, if the number of possible pairs is P, then the number of graphs is 2^P, since for each pair we have 2 choices (whether to put an edge or not) irrespective of what we did for the other pairs. Also, the number of pairs is just the number of ways we can choose two vertices to form a pair, which is $\binom{n}{2}$. Hence, the number of possible graphs is $2^{\binom{n}{2}}$. *3pts*

A large number of people counted the number of possible edges (the number of pairs) to be $(n-1)!$ in this problem.

(c) To construct an ordering of the numbers from 1 to $2n$, we proceed in the following way: we pick n slots out of the $2n$ possible ones, where we put the numbers from 1 to n (there is only one way to put them in these slots), and then we arrange the numbers from $n+1$ in the remaining slots. The number of ways to pick the slots for the numbers from 1 to n is $\binom{2n}{n}$ and the number of ways to arrange the numbers $n+1, \ldots, 2n$ in the remaining n slots is $n!$. Hence, the total number of possible orderings is *3pts*

$$\binom{2n}{n} \times n! = \frac{(2n)!}{n!n!} \times n! = \frac{(2n)!}{n!}$$

(d) If we remove the restriction that each committee must contain at least one member, then for each person we have 3 choices: whether to be in the Arts committee, to be in the Education committee or *to be in neither of them*. Hence, the total number of ways to choose the committees under this assumption is 3^n. *3pts*

However, now we need to subtract the number of ways in which we might be forming an empty committee. The number of ways in which we can form the committees so that the Arts committee has no members is 2^n (since then each person must be either in Education committee or in no committee). Similarly, the number of ways in which we can form an empty Education committee is 2^n. Subtracting these gives the number of ways as $3^n - 2 \cdot 2^n$. But we have subtracted the case when both committees are empty *twice* (since it is included both in cases when the Arts committee is empty or the Education committee). There is only one way in which both committees can be empty and we need to add this back once to take care of the double subtraction. Hence, the total number of ways is

$$3^n - 2 \cdot 2^n + 1$$

There were two common mistakes in this problem. One was to again ignoring the "at most" and taking each person to be in exactly one committee. The second mistake was to take the people as identical (!) and solve the problem using unordered balls and bins.

(e) We break this into two cases: either the social security number has *exactly 9 digits* or it has *exactly 8 digits*. Let us consider the case with exactly 9 digits first. For this to happen, all the digits in the social security number must be different. Thus, we have 10 choices for the first digit, 9 choices for the second digit, 8 choices for the third one and so on. The number of social security numbers with *exactly 9 digits* is *3pts*

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 10!$$

Now we count the number of social security numbers with *exactly 8 digits*. To do this, we can choose the 8 digits we want to use in $\binom{10}{8}$ ways. Once we have chosen the digits, we can pick the digit which will appear twice in $\binom{8}{1}$ ways. Now we are simply left with the problem of ordering 7 distinct digits and 2 copies of one digits. This can be done in $\frac{9!}{2!}$ ways. So the total number of SSNs with exactly 8 different digits is

$$\binom{10}{8} \times \binom{8}{1} \times \frac{9!}{2!} = \frac{10 \times 9}{2 \times 1} \times 8 \times \frac{9!}{2!} = 18 \times 10!$$

Hence, the total number of ways is $10! + 18 \times 10! = 19 \times 10!$.

A common error for this part was ignoring the issue of ordering and just counting the number of ways of choosing the digits. However, this does not suffice as different orderings of the digits definitely give different Social Security Numbers. Also, a large number of students counted to number of SSNs with 9 different digits as 9! instead of 10!.

4. Colorful Jelly Beans

(a) The correct way to think of this problem is in terms of 100 unlabeled balls (the jelly beans) in 3 labeled bins (the colors red, orange, yellow). Recall from class that we have a formula for this: $\binom{n+k-1}{k}$, where *3pts*

k is the number of balls and n is the number of bins. Therefore, the correct answer for this section is $N = \binom{102}{100} = 5151$.

A very common mistake in this section was people mixing up n and k in the formula; also, some people gave the answer 3^{100} - this is the case when both the balls and the bins are labeled (it would be a situation where it also matters which jelly beans are which color, and not just the total number of jelly beans of a given color).

(b) The probability that two jars of jelly beans are the same is $\frac{1}{n}$. There are many ways to arrive at this result; one way to think of this is given the configuration of the first jar of jelly beans, there is one choice out of n for the configuration of the second jar of jelly beans that causes it to be the same as the first jar of jelly beans. We can also use a counting argument: there are exactly n^2 different ways to assign configurations to the first two jars of jelly beans, and of those n^2 choices there n choices that result in them being the same. Therefore, the probability that they are the same is $\frac{n}{n^2} = \frac{1}{n}$. However, we are looking for the probability that two jars of jelly beans are different, which is $1 - \frac{1}{n}$. *2pts*

(c) We extend the result of part (b). The probability that the second jar of jelly beans is different from the first is $\left(1 - \frac{1}{n}\right)$, as described above. The probability that the third jar of jelly beans is different from the first two (given that the first two is different) is $\left(1 - \frac{2}{n}\right)$. We can continue this line of reasoning to the probability the the m^{th} jar of jelly beans is different from the first $m - 1$ (given that the first $m - 1$ are different) - it is $\left(1 - \frac{m-1}{n}\right)$. Therefore, the probability that all of the m jars of jelly beans are different is $\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right)$. *3pts*

(d) In this section we were looking for you to recognize that you are supposed to apply the result from hash tables (or the birthday paradox). The main point was to realize that in order to have the probability of collision be greater than $\frac{1}{2}$ that you would need m to be on the order of $\sqrt{N}$. Since $N = 5151$, m would have to be on the order of 100. Many people actually remembered the formula from class and arrived at the result that $m = 1.177\sqrt{N} \approx 84$. Full credit were given for such answers, but what we were looking for was the order of magnitude to be 100. *3pts*

Many people had the right approach to this section although this missed the answer to part (a). Full credit was given if they had the right approach with an incorrect value of N.

5. Learning and Betting

(a) Let A be the event that the good coin is picked and B be the event that the randomly chosen coin comes up Head. By the total probability rule, *3pts*

$$\Pr[B] = \Pr[B|A]\Pr[A] + \Pr[B|\bar{A}]\Pr[\bar{A}] = 0.55 \times 0.5 + 0.2 \times 0.5 = 0.375. \quad (1)$$

Since this probability is less than 0.5, it is not a good game to play.

(b) By Bayes' rule, *5pts*

$$\Pr[A|B] = \frac{\Pr[B|A]\Pr[A]}{\Pr[B]} = \frac{0.55 \times 0.5}{0.375} = 0.734,$$

which means that the conditional probability that I picked a good coin given I saw a Head is greater than 0.5, so I will keep this coin for my next flip.

Let C be the event that I will get a Head on my next flip (using the same coin). Applying the total probability rule to the event C under the condition B, we get:

$$\Pr[C|B] = \Pr[C|A, B]\Pr[A|B] + \Pr[C|\bar{A}, B]\Pr[\bar{A}|B] = 0.55 \times 0.734 + 0.2 \times 0.266 = 0.460. \quad (2)$$

Since this is still less than 0.5, I would not place a bet.

Many people argued this by saying that conditional on seeing a Head from the first flip, we are in a new sample space where the probability of the chosen coin being the good coin is $\Pr[A|B] = 0.734$ *instead of* $\Pr[A] = 0.5$ *in the calculation of* $\Pr[B]$ *in eqn (1). This is what the calculation in (2) formalizes but the informal argument earns a full score as well. Another solution is to go to the full sample space where the sample points are the triples* (x, y, z)*, where* $x = G$ *if the good coin was selected and* $x = B$ *otherwise,* $y = H$ *if the first flip is a Head and* $y = T$ *otherwise, and* $z = H$ *if the second flip is a Head and* $z = T$ *otherwise. The conditional probability* $\Pr[C|B]$ *can be explicitly calculated.*

(c) Let D be the event that two heads are observed. By the total probability rule, *5pts*

$$\Pr[D] = \Pr[D|A]\Pr[A] + \Pr[D|\bar{A}]\Pr[\bar{A}] = (0.55)^2 \times 0.5 + (0.2)^2 \times 0.5 = 0.17. \quad (3)$$

By Bayes' rule,

$$\Pr[A|D] = \frac{\Pr[D|A]\Pr[A]}{\Pr[D]} = \frac{(0.55)^2 \times 0.5}{0.17} = 0.88$$

So obviously we should stick to the same coin for the third flip. Let E be the event that the third flip is a Head. By total probability rule,

$$\Pr[E|D] = \Pr[E|A, D]\Pr[A|D] + \Pr[E|\bar{A}, D]\Pr[\bar{A}|D] = 0.55 \times 0.88 + 0.2 \times 0.12 = 0.509.$$

Now, it's worth a bet!.