

Midterm 1

6:00-8:00pm, 3 October

*Notes: There are **four** questions on this midterm. Answer each question part in the space below it, using the back of the sheet to continue your answer if necessary. If you need more space, use the blank sheet at the end. **None of the questions requires a very long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.** The approximate credit for each question part is shown in the margin (total 60 points). Points are not necessarily an indication of difficulty!*

Your Name:

Your Section:

For official use; please do not write below this line!

Q1	
Q2	
Q3	
Q4	
Total	

[exam starts on next page]

1. Quick Questions

(a) Write down the truth tables for (i) $P \Rightarrow Q$; (ii) $Q \Rightarrow P$; (iii) $P \Leftrightarrow Q$.

6pts

(b) Let $P(x)$ and $Q(y)$ be propositions about integers x and y respectively, and suppose you want to prove the theorem $(\exists x P(x)) \Rightarrow (\forall y Q(y))$. Which of the following proof strategies would be a valid way to proceed? [Note: There may be no valid strategies, or more than one valid strategy. You do *not* need to explain your answers.] 8pts

- (i) Assume that there is some y for which $Q(y)$ is false, and deduce that $P(x)$ is false for all x .
- (ii) Assume that $Q(y)$ is true for all y , and deduce that $P(x)$ is true for some x .
- (iii) Assume that, for some x and some y , $P(x)$ is true and $Q(y)$ is false, and deduce a contradiction.
- (iv) Assume that $P(x)$ is true for all x and $Q(y)$ is false for all y , and deduce a contradiction.

[continued on next page]

[Q1 continued]

- (c) Consider an instance of the Stable Marriage problem in which the men are $\{1, 2, 3, 4\}$, the women are $\{A, B, C, D\}$, and the preference lists are *4pts*

Men (1-4)	Women (A-B)
1: B A D C	A: 2 1 4 3
2: C A D B	B: 3 4 2 1
3: A C B D	C: 1 4 3 2
4: D C B A	D: 2 3 1 4

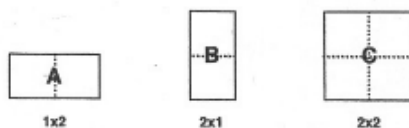
List the stable pairing given by the traditional propose-and-reject algorithm on this instance. [You need *not* show the execution of the algorithm.]

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- (d) Compute the multiplicative inverse of 10 modulo 743 using the extended GCD algorithm discussed in class. Show your working! Your answer should be an integer in the range $[0, 742]$. *6pts*

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2. A Tiling Problem

You are given three kinds of tiles, **A**, **B**, and **C**, of dimensions 1×2 , 2×1 , and 2×2 respectively (as shown in the figure below). Note that rotations of the tiles are not allowed, so tiles **A** and **B** are **not** the same!



Your goal is to tile a board of dimension $2 \times n$, and you are interested in the number of tiling configurations possible (using the three types of tiles given). The figure below shows an example of three different tiling configurations for a 2×5 (i.e. $n = 5$) board.



Let T_n denote the number of tiling configurations for a board of dimension $2 \times n$.

- (a) Compute T_1 and T_2 .

2pts

- (b) Explain carefully why $T_n = T_{n-1} + 2T_{n-2}$. [HINT: The configurations in the example given above for $n = 5$ may be helpful.]

4pts

- (c) Prove by induction that $T_n = \frac{2^{n+1} + (-1)^n}{3}$ for all $n \geq 1$. Show clearly the structure of your proof!

9pts

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3. Modular Arithmetic

Which of the following statements is true? In each case, if the statement is true give a brief explanation; if it is false, give a simple counterexample.

- (i) For all $a, b \in \mathbf{Z}$, $(a + b)^3 = a^3 + b^3 \pmod{3}$. [Note: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.] 3pts
- (ii) For all $a, b \in \mathbf{Z}$, $(a + b)^4 = a^4 + b^4 \pmod{4}$. [Note: $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.] 3pts
- (iii) For all $a, b \in \mathbf{Z}$, $(a + b)^5 = a^5 + b^5 \pmod{5}$. [Note: $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.] 3pts

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4. Stable Marriage

For each of the following properties, say whether the property is true or false. If the property is true, provide a *short* proof. If it is false, provide a *simple* counterexample. You may use without proof any facts about Stable Marriage covered in class provided they are clearly stated.

- (a) In a Stable Marriage instance with at least two men and two women, if man M and woman W each put each other at the top of their respective preference lists then there exists a stable pairing in which M and W are paired together. 4pts

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- (b) In a Stable Marriage instance with at least two men and two women, if man M and woman W each put each other at the bottom of their respective preference lists then there is *no* stable pairing in which M and W are paired together. 4pts

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- (c) In a Stable Marriage instance with at least two men and two women, it is possible that there exists an unstable pairing in which *every* unmatched man-woman pair (M, W) is a rogue couple. 4pts

[The end]