CS 70 Discrete Mathematics for CS
Fall 2001 Wagner Midterm 1

PRINT your name: ____

SIGN your name; ____

This exam is closed-book, closed-notes. One page of notes is permitted. Calculators are permitted. Do all your work on the pages of this examination.

You have 2 hours. There are 4 questions, of varying credit (50 points total). You should be able to finish all the questions, so avoid spending too long on any one question.

1. (12 pts.) Short-answer questions

Translate each of the following claims into symbolic form. For instance, a good translation of "*n* is either at least three or at most five" would be "$n \geq 3 \vee n \leq 5$."

Then, state whether the claim is true or false, and briefly justify your answer.

(a) [3 pts.] There is some natural number whose square root is not a natural number.

$\exists n \in \mathbf{N}, \sqrt{n} \notin \mathbf{N}$ (or: $\exists n \in \mathbf{N}, \forall k \in \mathbf{N}, k^2 \neq n$)

TRUE: $n = 2$ is an example, since we showed in class that $\sqrt{2}$ is not a natural number.

(b) [4 pts.] For every natural number *n*, one can find another natural number *m* that is strictly smaller than *n*.

$\forall n \in \mathbf{N}, \exists m \in \mathbf{N}, m < n$

FALSE: for $n = 0$, there is no $m \in \mathbf{N}$ with $m < 0$.

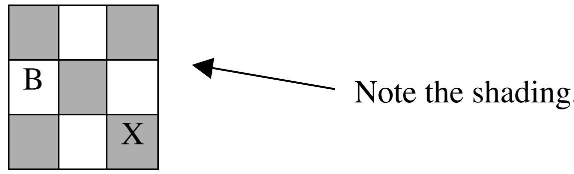
(c) [5 pts.] For each natural number *k* there is some lower bound ℓ so that $k^n \geq n!$ when $n \geq \ell$.

$\forall k \in \mathbf{N}, \exists \ell \in \mathbf{N}, \forall n \in \mathbf{N}, n \geq \ell \Rightarrow k^n \geq n!$

FALSE: A counterexample is $k = 1$, since $1^n < n!$ for all $n > 1$.

2. (12 pts.) Reachability

In chess, a bishop can move diagonally in any of the four directions. Consider a 3×3 board, with a bishop initially placed at the location marked 'B' (see below). Prove that it can never reach the square marked 'X'.



Let P denote the property (of a configuration) that the bishop is on a light-colored square.

Let $Q(n)$ denote the claim that, after any sequence of n moves, we end in a configuration satisfying P.

We prove by simple induction on n that $Q(n)$ holds $\forall n \geq 0$.

Base case: $Q(0)$ holds, since the initial configuration satisfies P.

Inductive step: We show $\forall n \geq 0,\ Q(n) \Rightarrow Q(n+1)$.

Pf: Fix n. Suppose $Q(n)$ holds (otherwise there is nothing to prove). Consider any sequence of n+1 moves. This can be broken into an initial segment of n moves, followed by a final move. After the first moves, P holds, since we assumed $Q(n)$. But now P must hold after the last move, too, since no single move can take the bishop from a light-colored to a dark-colored square. Thus $Q(n+1)$ holds, since our choice of n+1 moves was arbitrary. □

We've shown that, in every reachable configuration, the bishop is on a light-colored square; since 'X' is on a dark square, 'X' is unreachable, no matter how many moves we make. □

3. (16 pts.) Proof by induction

Let the sequence $a_0, a_1, a_2, \ldots$ be defined by the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \text{ for } n \geq 2 \text{ and } a_0 = 1, a_1 = 2.$$

Consider the following argument:

Theorem 1 $a_n \leq n + 2$ *for all* $n \geq 0$.

Proof: We use strong induction on *n*. The base cases $n = 0$ and $n = 1$ hold, since $a_0 = 1 \leq 0 + 2$ and $a_1 = 2 \leq 1 + 2$. Now if $a_i \leq i + 2$ for each $i = 0, 1, \ldots, n-1$, for some $n \geq 2$, then we have

$$a_n = 2a_{n-1} - \underline{a_{n-2}} \leq 2((n-1)+2) - \underline{((n-2)+2)} \leq 2n + 2 - n \leq n + 2,$$

which shows that $a_n \leq n + 2$ holds for all $n \geq 0$. □

(a) [6 pts.] Critique the above proof.

The problem is in the underlined step.

It is true that $a_{n-2} \leq (n-2) + 2$, but not valid to conclude that
$2a_{n-1} - a_{n-2} \leq 2a_{n-1} - ((n-2) + 2)$;
due to the negative sign, we must reverse the inequality.

(b) [10 pts.] Give a better proof of the theorem.

Claim: $a_n = n + 1$ for all $n \geq 0$.
Pf: By strong induction on *n*. Let $P(n)$ = "$a_n = n + 1$".
Base cases: $P(0)$ holds, since $a_0 = 1 = 0 + 1$.
$P(1)$ holds, since $a_1 = 2 = 1 + 1$.
Inductive step: We show $P(0) \wedge P(1) \wedge \cdots \wedge P(n-1) \Rightarrow P(n) \; \forall n \geq 2$.
Pf: Assume $a_i = i + 1$ for $i = 0, 1, \ldots, n-1$. Then

$$a_n = 2a_{n-2} - a_{n-2} = 2((n-1)+1) - ((n-2)+1)$$
$$= 2n - (n-1) = n + 1.$$

□

This shows that $a_n = n + 1 \; \forall n \geq 0$, from which the desired result ($a_n \leq n + 2 \; \forall n \geq 0$) follows.

The trick was to strengthen the hypothesis.

4. (10 pts.) Matchings

Recall that a *matching* on n boys and m girls is a pairing where each boy is married to exactly one girl and each girl is married to exactly one boy.

(c) [5 pts.] Let M be a stable matching on n boys and n girls where Alice is paired with Bob. Now Alice and Bob fly off the Bermuda on vacation. We are left with a matching, call it L, on the remaining n–1 boys and n–1 girls according to who is still paired up. Is L guaranteed to be a *stable* matching, if M is stable? Prove your answer.

YES. Assume not, i.e., we have an unstable pair in L:

A_1 ——— B_1 where A_1 prefers B_2 to B_1, and

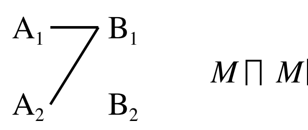
A_2 ——— B_2 B_2 prefers A_1 to A_2.

Then this is an unstable pair in M, contradicting the assumption of stability of M. Thus no unstable pair in L can exist, so L is stable, too.

(d) [5 pts.] If M, M' are two matchings, let $M \wedge M'$ denote the configuration where each girl is married to the better of her two partners in M and M' (according to that girl's preference list). Is $M \wedge M'$ guaranteed to be a matching? Prove your answer.
(Note that none of the matchings here are required to be stable.)

NO. Suppose A_1, A_2 both prefer B_1 to B_2.
A_1, A_2 are girls. Consider the following matchings:

Then $M \wedge M'$ is:



which is not a matching, since B_1 has two mates and B_2 has none.
So this is a counterexample.

Finished! You're done; this is the last page; there are no more questions.