CS 174: Combinatorics and Discrete Probability Spring 2016

Midterm 1

Tuesday, March 8

***Notes**: Please read all the questions carefully and ask me for any clarifications. Some questions are harder than others; if you are stuck, I suggest you move on to a different question and later on come back to the one on which you were stuck. Please write clearly and legibly. Good Luck!*

Name (4 points!):

SID:

Problem 1. *(Total 24 points)*
Let X_i be the outcome of rolling a fair 3 sided die, i.e. $Pr(X_i = j) = 1/3$ for $j \in 1, 2, 3$. Let $X = X_1 + X_2$, the outcome from rolling 2 fair 3-sided dice.

a) Compute $E[X]$.

b) Compute $Var[X]$.

c) Compute the moment generating function, $M_X(t)$.

d) Compute $Pr(X \geq 6)$.

e) Compute the Markov bound on $Pr(X \geq 6)$.

f) Compute the Chebyshev bound on $Pr(X \geq 6)$.

g) Derive a Chernoff bound on $Pr(X \geq 6)$.

Problem 2. *(Total 12 points)*

Consider a process that at fixed intervals either spawns 3 new processes or dies, with equal probabilities. Starting with one such process, is the expected total number of processes finite?

Problem 3. *(Total 12 points)*

Given a positive integer k and some $\mu > 0$, describe a random variable X with $E[X] = \mu$ that assumes only nonnegative values such that $Pr(X \geq k) = E[X]/k$.

Problem 4. *(Total 24 points)*

Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these bits in some order. Let Y_i be the exclusive-or of the i'th pair of bits, and let $Y = \sum_{i=1}^{m} Y_i$.

a) Show that each is 0 with probability 1/2 and 1 with probability 1/2.

b)Show that the Y_i are not mutually independent.

c) Show that they satisfy the property that $E[Y_i Y_j] = E[Y_i]E[Y_j]$ for all $i \neq j$.

d) Show that $Var[Y] = \sum_{i=1}^{n} Var[Y_i]$.

Problem 5. *(Total 24 points)*

Recall the randomized algorithm for computing the median in which we take a random (multi-set) subset R of the data set S to find the median, m. Let $|S| = n$. Consider a modified version of that algorithm where we choose $|R| = n^{1/2}$, and choose u to be the $3n^{1/2}/4$'th element of R and d the $n^{1/2}/4$'th element of R when sorted. Also recall the set $C = \{x \in S | d < x < u\}$. Other than these changes, the algorithm is unchanged from the book/lecture and I suggest that you use the same approaches as used therein. In parts (a) and (b) you can use Markov bounds to simplify the analysis, even though they are quite weak.

a) Bound the probability that $d > m$ and $u < m$.

b) Bound the probability that $|C| > n/2$, assuming that the previous check was successful.

c) Explain what the probabilities in parts (a) and (b) correspond to in the correctness of the algorithm. What is the expected running time of a single run of this modified version of the algorithm, assuming it succeeds?

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