Solutions for CS174 Midterm

1. (a) This is 10 Bernoulli trials. So the expected number of heads is $0.8 \times 10 = 8$.

   (b) In a random permutation, $1, 2, 3$ can have $3! = 6$ total number of possible configurations. There are only two possible configurations if we require 1 and 2 both come before 3 in the permutation. Each configuration is equally likely. So the probability in the question is $2/6 = 1/3$.

   (c) Because X and Y are two independent random variables, $\mathrm{E}[XY] = \mathrm{E}[X] \cdot \mathrm{E}[Y] = 4 \times 4 = 16$.

   (d) $\ln n$.

2. Because X has approximately a Poisson distribution with parameter $\lambda = 1$, $\mathrm{E}[X] = \mathrm{Var}[X] = 1$.

   (a) $\Pr[X \geq 5] \leq \frac{\mathrm{E}[X]}{5} = \frac{1}{5}$.

   (b) $\Pr[X \geq 5] \leq \Pr[|X - 1| \geq 4] \leq \frac{1}{16}$.

   (c) $\Pr[X \geq 5] = \Pr[X \geq (1 + 4) \cdot 1] < e^{-4}$, because $\delta = 4 < 2e - 1$.

   (d) This is a two-step argument. First of all, X has approximately a Poisson distribution. This does not mean that X is a sum of Poisson trials. But the *binomial* distribution is a sum of Bernoulli trials and is well-approximated by a Poisson distribution. In other words $X \approx Y \approx Z$ where X is the number of fixed points, Y is a Poisson r.v., and Z is a binomial r.v. Chernoff can be applied to Z.

3. Details of the derivation are in the lecture notes.

   (a) This is the coupon collector problem. $n \ln n$.

   (b) This is the birthday-paradox problem. $\sqrt{2n}$.

(c) Expected number of proposals a male makes $= \ln n$, expected rank of his final spouse $= \ln n$.

(d) Expected number of proposals a female receives $= \ln n$, expected rank of her final spouse $= \frac{n}{\ln n}$.

4. (a) $n \ln n + \Omega(n)$.

(b) Approximately, we can consider the process of generating random graphs of n vertices as throwing balls into n bins. Adding an edge is equivalent to throwing two balls into the bins. So when we add n edges, i.e., throw $2n$ balls, the expected number of empty bins is $n(1-\frac{1}{n})^{2n} = \frac{n}{e^2}$. An empty bin means a single vertex that touches no edges, and is therefore an isolated connected piece. So the expected number of connected pieces is at least $\frac{n}{e^2} + 1$.

(c) Let Y_i denote the number of edges we need to add to make the graph change from i connected pieces to $i-1$ connected pieces. From the lecture notes, the probability of adding an edge that can change the graph from i connected pieces to $i-1$ connected pieces is greater than or equal to $(i-1)/(n-1)$.

Let X_i denote the number of balls we need to throw into $n-1$ bins to change the number of empty bins from $i-1$ to $i-2$. So the probability of adding a ball that can change the number of empty bins from $i-1$ to $i-2$ is $(i-1)/(n-1)$. So $\Pr[X_i \geq k] \geq \Pr[Y_i \geq k]$. Because $\mathrm{E}[X_i] = \sum_k \Pr[X_i \geq k]$, and $\mathrm{E}[Y_i] = \sum_k \Pr[Y_i \geq k]$, $\mathrm{E}[X_i] \geq \mathrm{E}[Y_i]$. As we have computed in class, when we throw n balls into n bins, the expected number of empty bins is n/e. Let T_1 be the number of epochs the random graph goes through when we throw in n edges, which is the number of connected pieces in the graph. Let T_2 be the number of empty bins when we throw in n balls into n bins. Because $\mathrm{E}[X_i] \geq \mathrm{E}[Y_i]$, we can see that $\Pr[T_1 \geq k] \leq \Pr[T_2 \geq k]$, which implies $\mathrm{E}[T_1] \leq \mathrm{E}[T_2] = n/e$. So the upper bound is n/e.