CS174 Midterm November 6, 2003

Your name:

Answer each question in the space provided.

1. Give an upper bound on the probability that a Poisson random variable with mean 10 is greater than or equal to 20. Your answer can be given as a formula involving e and other constants.

2. Let $X_1, X_2, \cdots, X_n$ be events in a discrete sample space. Let $S_1 = \sum_{i=1}^{n} Pr(X_i)$ and $S_2 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr(X_i \cap X_j)$.

   Prove: $Pr(X_1 \cup X_2 \cup \cdots \cup X_n) \geq S_1 - S_2$.

   Hint: Consider a point in the sample space that lies in k of the events X_i. What is its contribution to each side of the inequality?

3. A family H of hash functions from a universe into a set of n hash addresses is called 3-universal if, for any 3 distinct keys x, y and z, the probability that $h(x) = h(y) = h(z)$, when h is drawn uniformly at random from H, is at most $\frac{1}{n^2}$.

   How large must n be to ensure that, when b keys are hashed into the table using a hash function drawn uniformly at random from a 3-universal family, the probability that there is a hash address which receives three or more keys is at most $1/6$?

4. Let π be a random permutation of the 2^n nodes of the n-dimensional unit hypercube. For each node u, a packet is sent from u to $\pi(u)$ using the left-to-right bit fixing algorithm.

   (a) What is the expected number of edges that the route from u to $\pi(u)$ traverses?

   (b) Consider any directed edge between neighbors in the hypercube. What is the expected number of packets traversing that edge during the routing of all the packets?

5. A *tournament* is a directed graph on n vertices such that, for every pair u, v of vertices, there is either an edge directed from u to v or an edge directed from v to u, but not both. A *Hamiltonian path* in a tournament is a directed path that starts at some vertex and visits each of the other vertices exactly once.

   Using the Probabilistic Method, prove that there exists a tournament containing at least $\frac{n!}{2^{n-1}}$ distinct Hamiltonian paths (not necessarily edge-disjoint). Carefully define the sample space and random variable you are using.

6. Let $X = X_1 + X_2 + \cdots + X_n$ where, for $i = 1, 2, \cdots, n$, $X_i = 1$ with probability p_i, and 0 with probability $1 - p_i$. The X_i's are not necessarily independent.

Prove: $E[X^2] = \sum_{i=1}^{n} p_i E[X|X_i = 1]$.