CS174   Midterm    November 6, 2003

Your name:

Answer each question in the space provided.

1. Give an upper bound on the probability that a Poisson random variable with mean 10 is greater than or equal to 20. Your answer can be given as a formula involving $e$ and other constants.
2. Let $X_1, X_2, \cdots, X_n$ be events in a discrete sample space. Let $S_1 = \sum_{i=1}^{n} Pr(X_i)$ and $S_2 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr(X_i \cap X_j)$.

Prove: $Pr(X_1 \cup X_2 \cup \cdots \cup X_n) \geq S_1 - S_2$.

Hint: Consider a point in the sample space that lies in $k$ of the events $X_i$. What is its contribution to each side of the inequality?
3. A family $H$ of hash functions from a universe into a set of $n$ hash addresses is called 3-universal if, for any 3 distinct keys $x$, $y$ and $z$, the probability that $h(x) = h(y) = h(z)$, when $h$ is drawn uniformly at random from $H$, is at most $\frac{1}{n^2}$.

How large must $n$ be to ensure that, when $b$ keys are hashed into the table using a hash function drawn uniformly at random from a 3-universal family, the probability that there is a hash address which receives three or more keys is at most $\frac{1}{6}$?
4. Let $\pi$ be a random permutation of the $2^n$ nodes of the $n$-dimensional unit hypercube. For each node $u$, a packet is sent from $u$ to $\pi(u)$ using the left-to-right bit fixing algorithm.

(a) What is the expected number of edges that the route from $u$ to $\pi(u)$ traverses?

(b) Consider any directed edge between neighbors in the hypercube. What is the expected number of packets traversing that edge during the routing of all the packets?
5. A *tournament* is a directed graph on $n$ vertices such that, for every pair $u, v$ of vertices, there is either an edge directed from $u$ to $v$ or an edge directed from $v$ to $u$, but not both. A *Hamiltonian path* in a tournament is a directed path that starts at some vertex and visits each of the other vertices exactly once.

Using the Probabilistic Method, prove that there exists a tournament containing at least $\frac{n!}{2^{n-1}}$ distinct Hamiltonian paths (not necessarily edge-disjoint). Carefully define the sample space and random variable you are using.
6. Let \( X = X_1 + X_2 + \cdots + X_n \) where, for \( i = 1, 2, \ldots, n \), \( X_i = 1 \) with probability \( p_i \), and \( 0 \) with probability \( 1 - p_i \). The \( X_i \)'s are not necessarily independent.

Prove: \( E[X^2] = \sum_{i=1}^{n} p_i E[X|X_i = 1] \).