Problem 1. (75 points) Consider the regular expression $R = (0 \cup 10 \cup 100)^*$.

a. Draw a nondeterministic finite automaton $N$ such that $L(N) = L(R)$. (You may take shortcuts that omit $\varepsilon$ transitions.)

b. Use the subset construction to draw a deterministic finite automaton $D$ such that $L(D) = L(N)$. (Label each state of $D$ with the corresponding set of states of $N$.)

c. Use the minimization algorithm followed by the quotient construction to draw a minimal DFA $M$ such that $L(M) = L(D)$. (Label each state of $M$ with the corresponding equivalence class of states of $D$.)

d. What is the index of $L(R)$? If the index of $L(R)$ is $k$, give $k$ words $w_1, \ldots, w_k$ such that no two of the words are $\equiv_{L(R)}$ equivalent. (You need not justify your answers.)

e. Give a regular expression $S$ such that $L(S) = [1]_{\equiv_{L(R)}}$. (Here $[1]_{\equiv_{L(R)}}$ stands for the $\equiv_{L(R)}$ equivalence class of the one-letter word 1.)

Problem 2. (75 points) For every language $A \subseteq \Sigma^*$, we define the following two languages:

\[\text{double}_1(A) = \{ x_1x_2x_3 \ldots x_{2n} \in \Sigma^* \mid n \geq 0 \text{ and } x_1x_2x_3 \ldots x_n \in A \}\]
\[\text{double}_2(A) = \{ x_1x_2x_3 \ldots x_{2n} \in \Sigma^* \mid n \geq 0 \text{ and } x_1x_3x_5 \ldots x_{2n-1} \in A \}\]

a. For $\Sigma = \{0, 1\}$ and $B = L(0^*)$, describe in words the two languages $\text{double}_1(B)$ and $\text{double}_2(B)$.

b. Which of the following two statements are true and which are false?

\begin{itemize}
  \item [S1] If $A$ is a regular language, then $\text{double}_1(A)$ is also regular.
  \item [S2] If $A$ is a regular language, then $\text{double}_2(A)$ is also regular.
\end{itemize}

To argue that one of these statements is true, you must define a finite automaton $M' = (Q', \Sigma', \delta', q_0', F')$ that accepts $\text{double}_1(A)$ from a given finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ that accepts $A$.

To argue that one of these statements is false, you must find a regular language $A$ and prove, using the pumping lemma, that $\text{double}_1(A)$ is not regular.