The last few questions are all or nothing. No partial credit will be given on these problems. So if you have an incomplete solution or a guess, please don’t bother to write it down.

Unless otherwise noted, each question is worth 20 points. Try to keep your answers succinct. Feel free to tear off the first 3 sheets, but please leave the rest stapled.

First, a few helpful theorems and definitions. Everything on this page may be useful.

**Lemma:** The Pumping Lemma:

If $L$ is regular

then $(\exists n)(\forall \epsilon \in L)$ such that $|\epsilon| \geq n$ $(\exists w \in L)$ such that $w = uvw$ and $|uw| \leq n$ and $|v| \geq 1$ $(\forall i) : \epsilon^i w \in L$

**Lemma:** The corollary of the Pumping Lemma:

If $(\forall n)(\exists \epsilon \in L)$ such that $|\epsilon| \geq n$ $(\forall w \in L)$ such that $w = uvw$ and $|uv| \leq n$ and $|v| \geq 1$ $(\exists i) : \epsilon^i w \not\in L$

then $L$ is not regular.

**Theorem:** Rice’s theorem: Let $L_P$ be the set of machines with property $P$. If $P$ is non-trivial, $L_P$ is undecidable. Further, $L_P$ is r.e. if and only if $P$ satisfies the following three conditions:

1. If $L \in P$ and $L \subseteq L'$ for some r.e. $L'$, then $L' \in P$.
2. If $L$ is an infinite language in $P$, then there exists a finite subset of $L$ in $P$.
3. The set of finite languages in $P$ is enumerable.

**3-SATISFIABILITY (3SAT)**

**INSTANCE:** A boolean formula, $F$, which is an AND of clauses where each clause is an OR of 3 literals.

**QUESTION:** Is $F$ satisfiable?

**3-DIMENSIONAL MATCHING (3DM)**

**INSTANCE:** A set $M \subseteq W \times X \times Y$, where $|W| = |X| = |Y| = \epsilon$ are disjoint sets.

**QUESTION:** Does $M$ contain a matching, $M' \subseteq M$, such that no two elements of $M'$ agree in any coordinate.

**VERTEX COVER (VC)**

**INSTANCE:** A graph $G$ and integer $K$

**QUESTION:** Is there a subset of $K$ vertices which cover all the edges?

**CLIQUE**

**INSTANCE:** A graph $G$ and integer $K$

**QUESTION:** Does the graph contain a clique (completely connected subgraph) of $K$ vertices?

**HAMILTONIAN CIRCUIT (HC)**

**INSTANCE:** A graph $G$

**QUESTION:** Is there a cycle through all the vertices of $G$?

**PARTITION**

**INSTANCE:** A finite set $A$ and a “size” $s(a) \in \mathbb{Z}^+$ for each $a \in A$.

**QUESTION:** Is there a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$$
1. (35 points) For the first 3 languages, give examples of strings in $L$ and not in $L$, and then determine if $L$ is regular. Prove your answer.

(a) $L = \{0^n \cdot 1^n : n \geq 0\}$

(b) $L = \{0^n 1^n : n \geq 0\}$

(c) $L = \{0^n 1^n : n \geq 0\}$

(d) Consider the following language:

$$LL^R = \{xy : x \in L \text{ and } y \in L^R\}$$

We know $L = \{0^n 1^n : n \geq 0\}$ is not regular. What is the language $LL^R$? Is it regular? Prove your answer.

(e) If $L$ is regular, is $LL^R$ also regular? Prove your answer.

2. (35 points) Which of the following are r.e.? Give a proof. (Hint: Any reductions can be done from $L_n$ by creating an $M'$ from $(M, w)$ which accepts either $\emptyset$ or $\Sigma^*$ depending on whether $M(w)$ rejects or accepts.)

(a) $L_{3M} = \{(M_1, M_2, M_3) : \text{At least two of the machines accept the same language.}\}$

(b) $L_{2M} = \{(M) : M(e) \text{ never moves past the } |Q|^n \text{ tape square}\}$. ($Q$ is the set of states of $M$.)

(c) $L = \{(M) : M(e) \text{ never moves past the } |Q|^n \text{ tape square}\}$. ($Q$ is the set of states of $M$.)

3. (35 points) Which of the following are non-trivial, that $L_1 \neq L_2$. (A language, $L$, is non-trivial if it's neither $\emptyset$ nor $\Sigma^*$. In other words, there is a string $x \in L_2$ and another string $x' \notin L_2$.)

(a) Show that if $L_1$ is recursive and $L_2$ is non-trivial, that $L_1 \cup L_2$. (A language, $L_2$, is non-trivial if it's neither $\emptyset$ nor $\Sigma^*$. In other words, there is a string $x \in L_2$ and another string $x' \notin L_2$.)

(b) Show that if $P=NP$, that any non-trivial language in $P$ is NP-complete.

4. Recall that PSPACE is the set of languages which can be accepted by a deterministic turing machine which, on input $w$, uses a polynomial in the length of $w$ tape squares. Prove $NP \subseteq PSPACE$.

5. Show DOMINATING SET is NP-complete:

**INSTANCE:** Given a graph $G = (V, E)$ and a positive integer $K$.

**QUESTION:** Is there a subset $V' \subseteq V$ of fewer than $K$ vertices which covers all vertices of $G$. (i.e., each vertex is either in $V'$ or adjacent to a vertex in $V'$.)

**Hint:** Reduce from VERTEX COVER.
6. Consider the BLOCK THE HOLES problem:

INSTANCE: Integers $n$ and $k$ and a deck of $n$ cards shaped as below with $k$ circles down the left and right sides. Some of the circles are punched out to make holes.

QUESTION: Is there a way to stack the cards, some face up and some face down, so that all the holes are covered (so no light would shine through.)

In the above example, $n = k = 3$ and the *'s mark the holes. It is a yes instance of BLOCK THE HOLES, since all holes can be blocked by turning cards 1 and 3 face up, and card 2 face down:

Use 3-SAT to show BLOCK THE HOLES is NP-complete. Hint: A card you may eventually use is one with holes down one side. The card will serve a role similar to the element representing FALSE in SET-SPLITTING.

7. (All or nothing) If $L$ is regular, are the following two languages also always regular? Prove each answer.
   (a) $L_1 = \{xy : x0y \in L \text{ and } |x| = |y|\}$
   (b) $L_2 = \{x0y : x \in L \text{ and } |x| = |y|\}$

8. (All or nothing) Use VERTEX COVER to show SHORTEST COMMON SUBSEQUENCE is NP-complete:

   INSTANCE: Finite alphabet $\Sigma$, finite set $R$ of strings from $\Sigma^*$, and a positive integer $K$.
   
   QUESTION: Is there a string $w \in \Sigma^*$ such that each string $x \in R$ is a subsequence of $w$.

   (For example, $R = \{ab, cb, ca, ac\}$ and $K = 4$ is a positive instance by choosing $w = abc$.)

9. (All or nothing) A language is defined to be in $D^P$ if it is the intersection of a language in NP with one in co-NP. In other words, $D^P$ is the set of languages which can be expressed as a set difference of two languages in NP.
   (a) Show UNIQUE-SAT is in $D^P$.
      
      INSTANCE: Boolean formula $F$
      
      QUESTION: Does $F$ have exactly one satisfying assignment of its variables?
   (b) Find a $D^P$-complete language. Provide a proof.