1. (35 Points) Prove that if $M$ is Turing machine with $q$ states, $w$ is an input of length $n$, and $M$ on input $w$ moves right on all the first $n + q + 1$ steps, then $M$ on input $w$ does not halt.
2. (35 Points) We use $\ell(x)$ to denote the length of a string $x$, and $K(x)$ to denote the smallest $k$ such that there exists a pair $(M, w)$ such that $\ell(M, w) \leq k$ and $M$ on input $w$ outputs $x$. We also define the language $R = \{ x : K(x) \geq \ell(x) \}$ of Kolmogorov random strings.

Prove that if $L \subseteq R$ is recognizable language, then $L$ must contain a finite number of strings. (Note that, in particular, this implies that $R$ is not recognizable.)

[Hint: prove that if $L \subseteq R$ is recognizable and infinite, then there is an algorithm that on input an integer $n$ outputs a string in $R$ of length at least $n$, and then prove that this leads to a contradiction.]
3. (30 Points) Prove that if $A, B$ are two languages in NP, then $A \cup B$ and $A \cap B$ are also in NP.