Problem #1 (20 points)

Problem #1a
Indicate an augmenting path in the following network by darkening the edges used along the path.

Problem #1b
Indicate below a maximum flow in the same network shown below by writing in those flow values that differ from part (a)
Problem #1c
Prove your flow is maximum by drawing a minimum cut in the graph below. (Naturally, the capacity of the cut should be identical to the max-flow you found above.)

Problem #2 (30 points)
Problem #2a
Compute $7^{129}$ mod 20. You may not use a calculator.

Problem #2b
Determine $x$ and $y$ such that $30x + 23y = \gcd(30, 23)$. You may not use a calculator.

Problem #2c
Add one edge to the following graph so that it has as few strongly connected components as possible.

![Graph Image]

**Problem #3 (30 points)**
Consider the problem of finding a perfect matching in a tree if one exists. Although your goal is to find a fast algorithm, you can receive 80% of the points for finding any polynomial time algorithm. Clarity is important, so I recommend solving part (d) separately so you don't lose points on parts a-c.

**Problem #3a**
Give a polynomial algorithm to find a perfect match in a tree if one exists.

**Problem #3b**
Analyze the running time of your algorithm as a function of $V$.

**Problem #3c**
Be sure to explain why your algorithm is correct; i.e. show that your algorithm gives a perfect matching if and only if one exists.

**Problem #3d**
(20%) Give an $O(V)$ or $O(V \log V)$ algorithm.

**Problem #4 (20 points)**
Let $G$ be a connected undirected graph. Consider the problem of finding a spanning tree of $G$ of minimum height. The height of a tree is measured as the maximum distance from its root to a leaf. (Your algorithm should choose both a root and a tree.)

**Problem #4a**
Give any polynomial time algorithm to solve the problem.
Problem #4b
Analyze the running time of your algorithm in terms of $E$ and $V$.

Problem #4c
Be sure to explain why your algorithm is correct if not obvious.