CS 70 SPRING 2008 — DISCUSSION #2

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1. Classic Strong Induction Problems

Exercise 1. Let the sequence $a_0, a_1, a_2, \ldots$ be defined by the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$ and $a_0 = 1, a_1 = 2$. Prove that $a_n \leq n + 2$ for all $n \geq 0$.

Solution:
We will prove an even stronger claim, namely that $a_n = n + 1$.

Proof. Base case: $a_0 = 1$, $a_1 = 2$. *Inductive Step:* Assume that if k is smaller than some n, then $a_k = k + 1$. Then we can infer that $a_{n+1} = 2a_n - a_{n-1} \Rightarrow a_{n+1} = 2(n+1) - n \Rightarrow a_{n+1} = n + 2$ and we are done.

□

Notice that if our inductive hypothesis is "Assume that if k is smaller than some n, then $a_k \leq k+2$". We cannot state $a_{n+1} \leq 2(n+2) - (n+1)$, why not?

Exercise 2. One day, Prof. Wagner decided to bring to class a giant chocolate bar with $n \times m$ pieces. He would like to share the chocolate with everyone in the class by giving one piece to each. To break the chocolate into pieces, he would first break the large bar into two, then choose a half-bar and break that into half again, and then he would repeatedly choose one contiguous block of chocolate and break it into two until there are only single pieces of chocolate left. Prove that no matter how you break the chocolate, it would always take $nm - 1$ number of moves to break down the entire bar into single pieces.

Solution:
Proof by induction over n, **the number of pieces in the giant chocolate bar**.
Base case: If $n = 1$, then clearly it takes 0 breaks.
Inductive Step: Suppose that if a chocolate bar has k pieces where $k \leq n$ for some n, then it takes $k - 1$ moves to break it down. Consider a chocolate with $n + 1$ pieces. One break will separate it into two blocks, each with less than $n + 1$ pieces. Hence, by inductive hypothesis, the number of breaks we need is $(n_1 - 1) + (n_2 - 1) + 1 = (n_1 + n_2) - 1$ where n_1 is the number of pieces in the first sub-block and n_2 number of pieces in the second one.

Exercise 3. The sequence of Fibonacci numbers is defined by: $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ if $n \geq 2$. Prove that any natural number can be represented as the sum of several **distinct** Fibonacci numbers.

Solution:
We run strong induction over n, the natural number that we want to decompose into different Fibonacci numbers.
Base case: $n = 1$. Clearly true.
Assume that the claim is true for all natural numbers smaller than $n - 1$. Consider the largest Fibonacci number $\leq n$, which we will denote F_k. In another words, $F_k + F_{k-1} \geq n$. From there, we know then by simple algebra that $n - F_k \leq F_{k-1}$. Hence, $n - F_k$ is the sum of different Fibonacci numbers by I.H. and furthermore, the sum cannot contain F_k so we see that n can also be written as a sum of different Fibonacci numbers.

2. Invariants

Exercise 4. Consider a regular 8×8 chessboard with the two white corners removed. Show that this board cannot be tiled by 2×1 dominoes.

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Solution:
The proof will be by contradiction. So suppose we can find such a tiling. Let B be the number of black squares covered by the dominoes, and let W be the number of white squares covered by the dominoes. Since each domino covers one black square and one white square we must have $W = B$. But the board has 32 black squares and 30 white square, contradicting the fact that the board is completely covered by dominoes.

Exercise 5. *(This problem was created by Gabriel Carroll)* We have k switches arranged in a row, and each switch points up, down, left, or right. Whenever three successive switches all point in different directions, all three may be simultaneously turned so as to point in the fourth direction. Prove that this operation cannot be repeated infinitely many times.

Solution:
Number the switches $1, 2, \ldots, k$. For any given configuration of switches, let the "height" of the configuration bet the product of all values of n for which switches $n-1$ and n point in the same direction (or 1 if there are no such n); this is always a positive integer. We claim that the height increases with every operation. Indeed, consider the operation in which switches $n, n+1, n+2$ are turned. Before this operation, switches $n-1$ and n may (or may not) have pointed in the same direction, as may $n+2$ and $n+3$; no other such pairs can be broken, so the height is divided by at most $n(n+3)$. However, the pairs $n, n+1$ and $n+1, n+2$ are created, multiplying the height by $(n+1)(n+2)$. Thus if h is the height of a configuration, then the new height $h^{'}$ after one move satisfies:

$$h^{'} \geq h \cdot \frac{(n+1)(n+2)}{n(n+3)} = h \cdot \frac{n^2+3n+2}{n^2+3n} > h.$$

So, as claimed, the height does increase at every step. Since the height is an increasing integer, and it cannot exceed $1 \times 2 \times \cdots \times k$, it can only increase finitely many times, and the result follows.

3. Well-Ordering

Exercise 6. Consider an infinite sheet of graph paper such that each square contains a natural number. Suppose that the number in each square is equal to the average of the numbers in the four neighboring squares. Prove that each square contains the same number.

Solution:
By the well-ordering principle, there must be some smallest number n which appears in the grid. Let S be a square which contains the number n. To conduct this proof, we need the following definition: we say that a square T is of distance d from S if we can find a chain of $d+1$ consecutive squares which begins at S and ends at T. Now let T be any square in the grid. We must show that T contains the number n. The proof will be by induction on d, the distance from T to S.
Base Case: The base case occurs when $d = 0$, namely when $T = S$. Then there is nothing to prove, since S contains the number n by definition.
Inductive hypothesis: Assume that each square of distance d from S contains the number n.
Inductive Step: Suppose T is of distance $d+1$ from S. Then one of the four squares adjacent to T, say U, is of distance d from S. By the inductive hypothesis, U contains the number n. We will be done if we can show the following claim:
Claim: Let U be a square which contains the number n. Then each of the four squares adjacent to U also contains the number n.
Proof: Suppose the four squares adjacent to U contain the numbers w, x, y, and z. The proof will be by contradiction, so assume that one of the four numbers (say w) is not equal to n. By definition we have $n \leq w, x, y, z$ and in fact we have $n < w$. Then

$$\frac{w+x+y+z}{4} > \frac{n+x+y+z}{4} \geq \frac{n+n+n+n}{4} = n.$$

But since each square is equal to the average of the squares around it, the LHS is equal to n, a contradiction. This proves the claim, and finishes the proof.

4. Stable Marriage

Exercise 7. Consider an instance of the stable marriage problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Does every stable solution S for this instance contain the pair (m, w)?

Solution:
Yes. The proof is by contradiction. Suppose there were a stable solution to the problem that did not include a marriage between m and w. Then, m would prefer w to his wife and w would prefer m to her husband. Thus, m and w would form a rogue couple, contradicting the stability of the solution.

Exercise 8. In a large group of n men and n women, Bob, one of the men, gets tipped off that he is the second-highest preference on every womans list. Bob is pretty happy to hear this. Assuming the traditional (male-optimal) algorithm, might Bob be in for a disappointment? In particular, is it possible that he will end up with the woman he prefers the least of all?

Solution:
Yes. Suppose Bob prefers Jane least of all, but that all other women and men form pairs of first-choice picks. In the first round of proposals, all men, including Bob, will propose to their first-choice women. However, Bob's favorite woman will accept her favorite man's proposal, rejecting Bob. After this first stage, all men other than Bob and all women other than Jane will be paired with their first-choice picks. Therefore, Bob will remain unpaired until he proposes to Jane. Jane will ultimately accept because her first-choice man rejected her. In this case, being the second-highest preference on every woman's list does not help Bob at all.