CS 70 SPRING 2008 — DISCUSSION #3

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1. Classic Strong Induction Problems

Exercise 1. Let the sequence $a_0, a_1, a_2, \ldots$ be defined by the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$ and $a_0 = 1, a_1 = 2$. Prove that $a_n \leq n + 2$ for all $n \geq 0$.

Answer:

Exercise 2. One day, Prof. Wagner decided to bring to class a giant chocolate bar with $n \times m$ pieces. He would like to share the chocolate with everyone in the class by giving one piece to each. To break the chocolate into pieces, he would first break the large bar into two, then choose a half-bar and break that into half again, and then he would repeatedly choose one contiguous block of chocolate and break it into two until there are only single pieces of chocolate left. Prove that no matter how you break the chocolate, it would always take $nm - 1$ number of moves to break down the entire bar into single pieces.

Answer:

Exercise 3. The sequence of Fibonacci numbers is defined by: $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ if $n \geq 2$. Prove that any natural number can be represented as the sum of several **distinct** Fibonacci numbers.

Answer:

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2. Invariants

Exercise 4. Consider a regular 8×8 chessboard with the two white corners removed. Show that this board cannot be tiled by 2×1 dominoes.

Answer:

Exercise 5. *(This problem was created by Gabriel Carroll)* We have k switches arranged in a row, and each switch points up, down, left, or right. Whenever three successive switches all point in different directions, all three may be simultaneously turned so as to point in the fourth direction. Prove that this operation cannot be repeated infinitely many times.

Answer:

3. Well-Ordering

Exercise 6. Consider an infinite sheet of graph paper such that each square contains a natural number. Suppose that the number in each square is equal to the average of the numbers in the four neighboring squares. Prove that each square contains the same number.

Answer:

4. Stable Marriage

A pair (M, W) is **unstable** if there exists another pair $(M*, W*)$ such that M prefers $W*$ over W and $W*$ prefers M over $M*$.

The traditional propose and reject algorithm:

(1) Every M proposes to the most desired W on their list
(2) If W has a previous mate $M*$ that she likes less than M, then W ditches $M*$ and accepts M
(3) If everyone has a mate, end. If M was ditched or rejected by W, he crosses W from the list
(4) Go to step 1.

Exercise 7. Consider an instance of the stable marriage problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Does every stable solution S for this instance contain the pair (m, w)?

Answer:

Exercise 8. In a large group of n men and n women, Bob, one of the men, gets tipped off that he is the second-highest preference on every womans list. Bob is pretty happy to hear this. Assuming the traditional (male-optimal) algorithm, might Bob be in for a disappointment? In particular, is it possible that he will end up with the woman he prefers the least of all?

Answer: