CS 70 SPRING 2008 — DISCUSSION #2

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1. Administrivia

(1) Course Information
- Office Hours have been decided. Please go if you have any questions about the material covered, trouble about the homework, or if you want to talk about other topics or life in general. Feel free to email the course staff to arrange alternate office hours if none of the ones provided fit your schedule.

2. Simple Inductions

Exercise 1. Prove that for all real number x not equal to 1,

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$$

Solutions:
We induct over n, the number of terms in the series.
Base case: Let $n = 0$, then we get $1 = \frac{x-1}{x-1}$.
Assume that the claim is true for $n-1$, then

$$\begin{aligned}\sum_{i=0}^{n} x^i &= \sum_{i=0}^{n-1} x^i + x^n \\ &= \frac{x^n-1}{x-1} + x^n \\ &= \frac{x^n-1}{x-1} + x^n\frac{x-1}{x-1} \\ &= \frac{x^{n+1}-1}{x-1}\end{aligned}$$

Exercise 2. Prove that for all natural number n,

$$2^{n-1} \geq n$$

Solutions:
We induct over n, the natural number in the equation.
Base case: $n = 0$. We see that $2^-1 = 1/2 \geq n$. If $n = 1$, we see that $2^0 = 1 \geq 1$. Assume that for some n, $2^{n-1} \geq n$. We can assume that $n \geq 1$, thus, $2^{n-1} + 2^{n-1} \geq n + n \geq n + 1$. Thus, $2 * 2^{n-1} \geq n + 1$ and that implies $2^n \geq n + 1$.

Date: February 8, 2008.

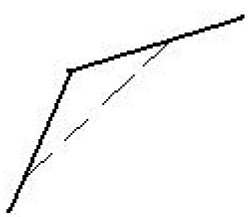
The authors gratefully acknowledge the TA's of CS70 Past for the use of their previous notes: Assane Gueye, Vahab Pournaghshband, Alex Fabrikant, Chris Crutchfield, Amir Kamil, David Garmire, Lorenzo Orecchia, and Ben Rubinstein. Their notes form the basis for this handout.

3. Some Trickier Inductions

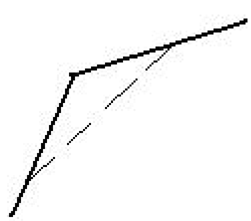
Exercise 3. Prove that if a number n comprises of just 3^k digits of "1"s (i.e. $n = \underbrace{111...111}_{3^k}$) for all natural number k greater than 0, then n is divsible by 3^k. For example, 111 is divisible by 3, 111,111,111 is divisible by 9, and so forth. (**hint:** try dividing 111,111,111 by 111 and remember that a number is divisble by 3 iff the sum of all of the digits of the number is divisible by 3.)

Solutions:
We induct over k, where a number comprises of just 3^k digits of "1"s.
Base case: Let $k = 1$, then 111 is divisible by 3 since $1 + 1 + 1 = 3$.
Assume that the claim is true for $k - 1$.
Suppose that the number n has 3^k digits of "1"s, then n is divisible by the number that has just 3^{k-1} digits of "1"s. More explicity, $\underbrace{111...111}_{3^k} / \underbrace{111...111}_{3^{k-1}} = 100...100...100....$ In this equation, which is of the form $a/b = c$, we know that b is divisible by 3^{k-1} by inductive hypothesis and c is divisible by 3 since its digits when added together is 3. Thus, we know that a is divisible by 3^k.

Exercise 4. Prove that, for any natural number n such that $n \geq 3$, there exists a convex n-gon (a convex n-gon is a n-sided polygon where each interior angle is less than 180 degrees) with exactly 3 acute angles. (**hint:** consider the figure below)



Solutions:
We induct over n, the number of sides of the polygon.
Base case: $n = 3$. The equilateral triangle has exactly 3 acute angles.
Assume that the claim is true for $n - 1$.
Consider a convex $(n - 1)$-gon with exactly 3 acute angles (one has to exist by I.H.). We pick an obtuse angle and saw it off from the polygon as shown below. This clearly gives us a n-gon with



exactly 3 acute angles.

Exercise 5. A group of people with assorted eye colors live on an island. They have all taken CS 70 and thus are perfect logicians - if a conclusion can be logically deduced, they will do it instantly. No one knows the color of their eyes. Every night at midnight, a ferry stops at the island. If anyone has figured out the color of their own eyes, they must leave the island that midnight.

On this island there are 100 blue-eyed people and 100 brown-eyed people, and the Guru (who has green eyes). One day, the Guru assembles everyone on the island and speaks for the first time in all their endless years on the island. Standing before them, she says the following:

"I can see someone who has blue eyes."

Use induction to show that on the 100th day, all 100 blue-eyed people will leave the island at once.

Solutions:
We first like to prove that if there are n blue-eyed people on the island, then they would all leave the island on n-th day. If we can prove that, then surely we prove the original proposition.
We induct over n, the number of blue-eyed people.
Base case: $n = 1$. Well, the one lone blue-eyer would look at everyone else, realize that he/she is the only person with blue-eye and hence leave on the first day.
Now assume that n blue-eyed people will all leave on the n-th day. Now suppose there are $n + 1$ blue-eyed people on the island. Let Min be a blue-eyed person. He would then look around the island and see n blue-eyed people. If he himself has brown-eyed people, then all n people would leave on the n-th day by inductive hypothesis. But this won't happened because every other blue-eyed people would see n blue-eyes also. Thus, on the $n+1$ day, upon seeing the island still populated with blue-eyers, Min would realize that he himself has blue-eye and thus leave the island on that day. Since the situtation is symmetric, every other blue-eyed person would reach the same realization and all leave the island on the $n + 1$ day.

4. Variations on Induction

Exercise 6. Suppose you know that $P(1)$ is true, and that $\forall k \geq 1, P(k) \Rightarrow P(2k)$. Use induction to show that $P(n)$ is true whenever n is a power of 2.

Solutions:
Let $n = 2^k$, we induct over k.
Base case: $k = 0$, then $P(2^k) = P(1)$ is true.
Assume that $P(2^{k-1})$ is true. Since we know that $P(n) \Rightarrow P(2n)$, $P(2^{k-1}) \Rightarrow P(2^k)$, and we conclude that $P(2^k)$ is true.

Exercise 7. Prove that $2^{m+n-2} \geq mn$ for all positive integers m, n.

Solutions:
We induct over m.
Base case: $m = 1$, we must show that $2^{n-1} \geq n$. To prove this, we induct over n.

Base case: let $n = 1$. Then $2 \geq 1$ certainly.
Assume that $2^{n-2} \geq n - 1$. Then $2(2^{n-2}) \geq (n - 1) + 1$ certainly $\Rightarrow 2^{n-1} \geq n$.

Assume that $2^{m+n-2} \geq mn$ is true. We know that $mn \geq n$, thus, $2(2^{m+n-2}) \geq mn + n \Rightarrow 2^{(m+1)+n-2} \geq n(m + 1)$.

5. Proofreading

Consider the following proofs and assign them either a grade of "A" or "F". Be sure to explain your rationale, students don't like to receive unexplicable "F"s.

Exercise 8. Claim: Prove that, for any natural number n, if I have n square-shaped paper sheets, that I can cut them into pieces in some way and then recombine the pieces into one large square-shaped sheet of paper.

Proof. Base Case: $n = 1$, the theorem is clearly true.
Assume that the claim is true for $n - 1$ number of squares.
Now, consider n squares. By inductive hypothesis, we can cut and combine any 2 of the squares to form one large square. And now we have a total of $n - 1$ squares and by inductive hypothesis again, we can cut and combine all of them to get our one large square. □

Solutions:
The proof fails when we're trying to go from $P(1) \Rightarrow P(2)$. We cannot combine 2 of the squares we have to form one large square if we only have 2 squares.