CS 70 SPRING 2008 — DISCUSSION #1

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1. Administrivia

(1) Course Information
- Office Hours have been decided. Please go if you have any questions about the material covered, trouble about the homework, or if you want to talk about other topics or life in general. Feel free to email the course staff to arrange alternate office hours if none of the ones provided fit your schedule.

2. Simple Inductions

Exercise 1. Prove that for all real number x not equal to 1,

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

Answer:

Exercise 2. Prove that for all natural number n,

$$2^{n-1} \geq n$$

Answer:

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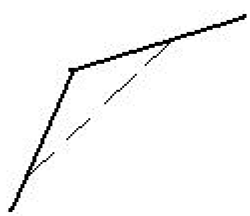
3. Some Trickier Inductions

Exercise 3. Prove that if a number n comprises of just 3^k digits of "1"s (i.e. $n = \underbrace{111...111}_{3^k}$) for all natural number k greater than 0, then n is divsible by 3^k. For example, 111 is divisible by 3, 111,111,111 is divisible by 9, and so forth. (**hint:** try dividing 111,111,111 by 111 and remember that a number is divisble by 3 iff the sum of all of the digits of the number is divisible by 3.)

Answer:

Exercise 4. Prove that, for any natural number n such that $n \geq 3$, there exists a convex n-gon (a convex n-gon is a n-sided polygon where each interior angle is less than 180 degrees) with exactly 3 acute angles. (**hint:** consider the figure below)



Answer:

Exercise 5. A group of people with assorted eye colors live on an island. They have all taken CS 70 and thus are perfect logicians - if a conclusion can be logically deduced, they will do it instantly. No one knows the color of their eyes. Every night at midnight, a ferry stops at the island. If anyone has figured out the color of their own eyes, they must leave the island that midnight.

On this island there are 100 blue-eyed people and 100 brown-eyed people, and the Guru (who has green eyes). One day, the Guru assembles everyone on the island and speaks for the first time in all their endless years on the island. Standing before them, she says the following:

"I can see someone who has blue eyes."

Use induction to show that on the 100th day, all 100 blue-eyed people will leave the island at once.

Answer:

4. Variations on Induction

Exercise 6. Suppose you know that $P(1)$ is true, and that $\forall k \geq 1, P(k) \Rightarrow P(2k)$. Use induction to show that $P(n)$ is true whenever n is a power of 2.

Answer:

Exercise 7. Prove that $2^{m+n-2} \geq mn$ for all positive integers m, n.

Answer:

5. Proofreading

Consider the following proofs and assign them either a grade of "A" or "F". Be sure to explain your rationale, students don't like to receive unexplicable "F"s.

Exercise 8. Claim: Prove that, for any natural number n, if I have n square-shaped paper sheets, that I can cut them into pieces in some way and then recombine the pieces into one large square-shaped sheet of paper.

Proof. Base Case: $n = 1$, the theorem is clearly true.
Assume that the claim is true for $n - 1$ number of squares.
Now, consider n squares. By inductive hypothesis, we can cut and combine any 2 of the squares to form one large square. And now we have a total of $n - 1$ squares and by inductive hypothesis again, we can cut and combine all of them to get our one large square. □

Answer: