1 An Inductive Proof

Suppose we have the following recursive rule for generating a String of 0s and 1s:

A $\rightarrow$ AAA or 0 or 1

To generate a string, we would start by writing a variable A and then make substitutions that the rule allows. We finish when our string is entirely made of 0s and 1s. For example:
A $\rightarrow$ AAA $\rightarrow$ AAA10 $\rightarrow$ 11AAA10 $\rightarrow$ 1100110.
Prove that the rule above can only generate binary strings of odd length.

2 Midterm Spring 2006

Prove that $\forall n \in \mathbb{N} 14^n - 1$ is divisible by 13.

3 Midterm Spring 2006

Evaluate $(\frac{6}{5})^{25} \mod 11$.

4 Rejected Midterm Question

Let x, y be integers in the equation

$$(3x + 7y)(9x + 6y) = 3$$

Find a solution or prove that none exists.

5 A Fundamental Question

Suppose $ax \equiv 1 \pmod{y}$. First, prove that the multiplicative inverse of $y \pmod{x}$ exists, then find the multiplicative inverse of $y \pmod{x}$ and express it in term of a, x, y.

6 Midterm Fall 2001

Let M be a stable matching on n boys and n girls where Alice is paired with Bob. Now Alice and Bob fly off the Bermuda on vacation. We are left with a matching, call it L, on the remaining n1 boys and n1 girls according to who is still paired up. Is L guaranteed to be a stable matching, if M is stable? Prove your answer.

7 Yet Another Fundamental Question

Supose that your calculator is broken and only 5 keys works: $0, 1, 2, \times, x^2$ where x^2 is the square key. Prove that your calculator cannot generate all possible even numbers.

8 Midterm Spring 2007

Consider the following two variants of induction.

1. Let P be a property of positive integers, and suppose you have proved that

   (a) $P(1)$ is true;

   (b) For every $n \geq 1$, $P(n) \Longleftrightarrow P(n+3)$

   (c) For every $n \geq 1$, $P(n) \Longleftrightarrow P(n+5)$

   Does it follow that $P(n)$ is true for every $n \geq 1$? Either prove that, or provide a counterexample. (A counterexample is a property P that is false for some $n \geq 1$, even though it satisfies properties (i), (ii), (iii).)

2. Let P be a property of positive integers, and suppose you have proved that

   (a) $P(1)$ is true;

   (b) For every $n \geq 1$, $P(n) \Longleftrightarrow P(n+4)$

   (c) For every $n \geq 1$, $P(n) \Longleftrightarrow P(n+6)$

   Does it following that $P(n)$ is true for every $n \geq 1$. Either prove that or show counterexample.

9 Midterm Spring 2006

For each of the following, indicate whether the two statements are equivalent.

1. $\forall x.(P(x) \vee Q(x))$ and $(\forall x.P(x)) \vee (\forall x.Q(x))$
2. $\forall x.(P(x) \wedge Q(x))$ and $(\forall x.P(x)) \wedge (\forall x.Q(x))$
3. $\forall x.\neg P(x)$ and $\neg \forall x.P(x)$

The famous French Mathematician Fermat believed (falsely) that he had proved the following theorem: $\forall n \in \mathbb{N}$ if $n = 2^{2^k} + 1, k \in \mathbb{N}$ then n is prime.

1. You have decided to prove this statement by contraposition. What statement would you be trying to prove.
2. You get confused and try to prove the converse instead. What statement would you prove in this case.

10 Midterm Fall 2006

Solve for x and y:

$$\begin{aligned} 3x + 5y &= 2 \pmod{19} \\ 7x + 3y &= 8 \pmod{19} \end{aligned}$$

11 Midterm Fall 2001

Translate each of the following claims into symbolic form. For instance, a good translation of "n is either at least three or at most five" would be "$n \geq 3 \vee n \leq 5$."

Then, state whether the claim is true or false, and briefly justify your answer.

1. There is some natural number whose square root is not a natural number.
2. For every natural number n, one can find another natural number m that is strictly smaller than n.
3. For each natural number k there is some lower bound l so that $k^n \geq n!$ when $n \geq l$.