MATH 202A — LECTURE NOTES FOR NOV 23, 2005

PROFESSOR DONALD SARASON

1. Lebesgue Integral

Lemma 1.1 (9.1)**.** *Let $f : R \to \mathbb{R}$ be Riemann integrable. Fix $\varepsilon > 0$. Then there exists $g : \mathbb{R}^N \to \mathbb{R}$, $h : \mathbb{R}^N \to \mathbb{R}$ continuous, $h|_R \le f \le g|_R$, and $\int_R (g-h)d\lambda_N < \varepsilon$.*

Proof. In the first case, let $f = \chi_J$, $J \subseteq R$ a closed rectangle. Choose an open rectangle $J_0 \supseteq J$ and a closed rectangle $J_{00} \subseteq \operatorname{int}(J)$ such that $\lambda_N(J_0 \setminus J_{00}) < \varepsilon$. For g, take a continuous function in $[0,1]$ such that $g = 1$ on R and $g = 0$ on $\mathbb{R}^N \setminus J_0$. For h, take a continuous function in $[0,1]$ such that $h = 1$ on J_{00} and $h = 0$ on $\mathbb{R}^N \setminus \operatorname{int}(J)$. Easy to see g and h meet the requirements.

In the second case, we suppose without loss of generality that $f \ge 0$. Take a partition P of R such that $U(f,P) - L(f,P) < \varepsilon/3$. Let k be the number of rectangles in the partition P. By step 1, for each $J \in P$, there exists a continuous $g_J : \mathbb{R}^N \to \mathbb{R}$, $h_J : \mathbb{R}^N \to \mathbb{R}$ such that $h_J|_R \le \chi_J \le g_J|_R$ and

$$\int_R (g_J - h_J)d\lambda_N < \frac{\varepsilon}{3kM},$$

where $M = \sup\{\, f(x) \mid x \in R \,\}$. Let $g = \sum_{J\in P} M_J g_J$, $h = \sum_{J\in P} m_J \chi_J$, g and h are continuous, $h \le f \le g$.

$$\begin{aligned}
\int_R (g-h)d\lambda_N &= \sum_{J\in P}\left(M_J\int_R g_J d\lambda_N - m_J\int_R h_J d\lambda_N\right)\\
&= \sum_{J\in P} M_J\left(\int_R g_J d\lambda_N - \lambda_N(J)\right) + \sum_{J\in P} m_J\left(\lambda_N(J) - \int_R h_J d\lambda_N\right)\\
&\quad +U(f,P) - L(f,P)\\
&\le Mk\left(\frac{\varepsilon}{3kM}\right) + Mk\left(\frac{\varepsilon}{3kM}\right) + \frac{\varepsilon}{3}\\
&= \varepsilon
\end{aligned}$$

□

Theorem 1.2 (9.2 — Lebesgue)**.** *Let $R \subseteq \mathbb{R}^N$ be a rectangle, $f : R \to \mathbb{R}$ bounded. Then f is Riemann integrable over R if and only if f is continuous almost everywhere.*

Proof. Assume f is Riemann integrable over R. By the lemma, there exists sequences $(g_n)_1^\infty$, $(h_n)_1^\infty$ of continuous functions on $\mathbb{R}^N$, $h_n \le f \le g_n$ on R, such that $\int_R g_n d\lambda_N \to \int_R f d\lambda_N$, $\int_R h_n d\lambda_N \to \int_R f d\lambda_N$. Without loss of generality, $g_{n+1} \le g_n$, $h_{n+1} \ge h_n$ for all n. Let $g = \lim g_n$, $h = \lim h_n$. Then $h \le f \le g$ on R, and $\int_R (g-h)d\lambda_N = 0$ because $\int_R g d\lambda_N = \int_R f d\lambda_N = \int_R hf d\lambda_N$ by MCT.

Scribed by Chris Crutchfield — Last updated November 23, 2005.

Therefore $g - h = 0$ almost everywhere on R. Thus $f = g = h$ almost everywhere on R.

Since g is the pointwise limit of a monotonically increasing sequence of continuous functions, it is uper semicontinuous ($g(x_0) \geq \lim_{x\to x_0} \sup g(x)$). Similarly, h is lower semicontinuous ($h(x_0) = \lim_{x\to x_0} \inf h(x)$). If $f(x_0) = g(x_0) = h(x_0)$, the nf is both upper semicontinuous and lower semicontinuous at x_0, hence continuous at x_0. This happens almost everywhere on R, so f is continuous almost everywhere on R.

Now assume f is continuous almost everywhere on R. We want to show that it is Riemann integrable. Let E be the set of discontinuities of f. Fix $\varepsilon > 0$. Take open $G \supseteq E$ such that $\lambda_N(G) < \varepsilon$. Let $K = R \setminus G$, a compact set, and f is continuous at each point of K. For each $x \in K$, there exists $\delta_K > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in R$ such that $\|y - x\| < \delta_x$. The balls $B_{\delta_x/2}(x)$ for $x \in K$ cover K, so there exists $x_1, \ldots, x_l \in K$ such that $K \subseteq \bigcup_{j=1}^{l} B_{\delta_{x_j}/2}(x_j)$. Let $\delta = \min\{\delta_{x_1}/2, \ldots, \delta_{x_l}/2\}$.

We assert that if $x \in K$, $y \in R$, $\|y-x\| < \delta$, then $|f(y)-f(x)| < 2\varepsilon$. In fact, there exists j such that $x \in B_{\delta_{x_j}/2}(x_j)$. Then $y \in B_{\delta_{x_j}/2}(x_j)$. Hence $|f(x) - f(x_j)| < \varepsilon$, $|f(y) - f(x_j)| < \varepsilon$, so $|f(x) - f(y)| < 2\varepsilon$.

Let P be a partition of R whose subrectangles all have diameter less than δ. Let P' be the rectangles of P that intersect K, P'' the rest of the rectangles in P. If $J \in P'$, $M_J - m_J \leq 2\varepsilon$ (see above).

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{J\in P} (M_J - m_J)\lambda_N(J) \\
&\leq 2\varepsilon\lambda_N(R) + (M - m) \sum_{J \in P''} \lambda_N(J) \\
&\leq 2\varepsilon\lambda_N(R) + (M - m)\lambda_N(G) \\
&\leq \varepsilon(2\lambda_N(R) + M - m)
\end{aligned}$$

Therefore f is Riemann integrable. □