# MATH 202A — LECTURE NOTES FOR NOVEMBER 21, 2005

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Last time we saw that the Riemann integral is really a Lebesgue integral.

## 1. Review of Riemann Integral

This will be a 104-style review, not 1A-style.

**Definition 1.1.** A rectangle in $\mathbb{R}^N$ is a product of compact integrals, i.e. a set of the form

$$R = \prod_{i=1}^{N} [a_i, b_i].$$

A partition of $R$ is a product

$$P = \prod_{i=1}^{N} P_i,$$

where $P_i$ is a partition of $[a_i, b_i]$ for each $i$, meaning

$$P_i = \{a_{i_0}, \dots, a_{i_{s_i}} : a_i = a_{i_0} < \cdots < a_{i_{s_i}} = b_i\}.$$

A subrectangle of $P$ is one of the rectangles

$$J_{j_1,\dots,j_N} = \prod_{i=1}^{N} [a_{i_{(j_i-1)}}, a_{i_{j_i}}],$$

where $j_i \in \{1, \dots, s_i\}$. The union of the subrectangles of $P$ is $R$. The union is disjoint to within a null set (Is this a typo?). We write $J \in P$ if $J$ is a subrectangle of $P$.

Let $f : R \to \mathbb{R}$ be bounded. Let $M = \sup\{f(x) : x \in R\}, m = \inf\{f(x) : x \in R\}$. Let $P$ be a partition of $R$. For a subrectangle $J \in P$, let $M_J = \sup\{f(x) : x \in J\}, m_J = \inf\{f(x) : x \in J\}$.

Let $U(f, P) = \sum_{J \in P} M_J \lambda_N(J) \le M\lambda_N(R)$ upper sum of $f$ over $P$.

Let $m\lambda_N(R) \le L(f, P) = \sum_{J \in P} m_J \lambda_N(J)$ lower sum of $f$ over $P$.

*Remark* 1.2. If $P_1, P_2$ are partitions, $P_2$ is a refinement of $P_1$ ($P_1 \subset P_2$), then $U(f, P_1) \ge U(f, P_2)$ and $L(f, P_1) \le L(f, P_2)$.

Now, if $P_1, P_2$ partitions of $R$, $P$ a common refinement, then

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2).$$

**Definition 1.3.** The upper Darboux integral of $f$ on $R$ is

$$\overline{\int_R} f = \inf_P U(f, P).$$

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The lower Darboux integral is

$$\underline{\int}_R f = \sup_P L(f, P).$$

We have

$$\underline{\int} f \leq \overline{\int} f.$$

$f$ is Riemann integrable if

$$\underline{\int} f = \overline{\int} f.$$

Then the common value is denoted $\int_R f(x)dx$.

**Theorem 1.4.** *If $f$ is Riemann integrable over $R$ then $f$ is Lebesgue integrable over $R$.*

*Proof.* Let $f$ be Riemann integrable.

Take a sequence $P_1, P_2, \ldots$ of partitions of $R$ s.t. $P_{n+1}$ is a partition of $P_n$ for each $n$, and

$$U(f, P_n) \to \int_R f(x)dx \to L(f, P_n) \to \int_R f(x)dx.$$

(As $n \to \infty$ I guess.)

Let

$$g_n = \sum_{J \in P_n} M_J \chi_J$$

and

$$h_n = \sum_{J \in R} m_J \chi_J$$

The $\chi$'s are characteristic functions. Then $h_n \leq g_n, g_{n+1} \leq g_n, h_{n+1} \geq h_n, h_n \leq f \leq g_n$ almost everywhere.

So ...

$$U(f, P_n) = \int_R g_n d\lambda_N \text{ and } L(f, P_n) = \int_R h_n d\lambda_N.$$

Let $g = \lim_{n \to \infty} g_n$ , $h = \lim h_n$. By Monotone Convergence Theorem, $g, h$ integrable and

$$\int_R g d\lambda_N = \int_R h d\lambda_N = \int_R f(x)dx$$

Now $\int_R (f - h)d\lambda = 0$. Since $g - h \geq 0$, $g = h = f$ almost everywhere. Therefore $f$ is Lebesgue integrable and $\int_R f d\lambda_N = \int_R f(x)dx$. □

**Theorem 1.5.** *Lebesgue's theorem. A bounded function $f : R \to \mathbb{R}$ is Riemann integrable iff it is continuous almost everywhere.*