MATH 202A — LECTURE NOTES FOR OCT 28, 2005

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1. Lebesgue Measure, etc.

Let λ be the Lebesgue measure on $\mathbb{R}$ and λ_N be the Lebesgue measure on $\mathbb{R}^N$. All Borel sets are Lebesgue measurable, and all Lebesgue null sets are Lebesgue measurable.

By Prop 4(b), the σ-algebra of Lebesgue measurable sets is the σ-algebra generated by the Borel sets and the null sets.

Example 1.1. Some examples of Null sets: countable sets, the cantor set in $\mathbb{R}$, lines in $\mathbb{R}^2$.

Remark 1.2 (Immediate Properties of λ_N). λ_N has the following properties:

(1) Translation invariance — If $E \subseteq \mathbb{R}^N$ is measurable and $x_0 \in \mathbb{R}^N$, then $E + x_0$ is measurable and $\lambda_N(E + x_0) = \lambda_N(E)$.

(2) If $r > 0$ and E is measurable, then $rE = \{\, re \mid e \in E \,\}$ is measurable and $\lambda_N(rE) = r^N \lambda_N(E)$.

Definition 1.3 (G_δ and F_σ Sets)**.** A set in a topological space is a G_δ if it is a countable intersection of open sets.

A set in a topological space is a F_σ if it is a countable union of closed sets.

Lemma 1.4 (5.1)**.** *Let $E \subseteq \mathbb{R}^N$ be measurable, $\lambda_N(E) < \infty$, $\varepsilon > 0$. Then there exists an open set G such that $E \subseteq G$ and $\lambda_N(G) < \lambda_N(E) + \varepsilon$.*

Proof. Take cells $C_1, C_2, \ldots$ such that $E \subseteq \bigcup_1^\infty C_n$ and $\sum_1^\infty \lambda_N(C_n) < \lambda_N(E) + \varepsilon/2$. For each n, take an open rectangle $R_n \supseteq C_n$ such that $\lambda_N(R_n) < \lambda_N(C_n) + \varepsilon/2^{n+1}$. Let $G = \bigcup_1^\infty R_n \supseteq E$.

$$\lambda_N(G) \le \sum_{n=1}^{\infty} \lambda_N(R_n) < \sum_{n=1}^{\infty} \left(\lambda_N(C_n) + \frac{\varepsilon}{2^{n+1}} \right) < \lambda_N(E) + \varepsilon$$

□

Proposition 1.5 (5.1)**.** *Let $E \subseteq \mathbb{R}^N$ be Lebesgue measurable. Then, given $\varepsilon > 0$, there exists open G, closed F such that $F \subseteq E \subseteq G$ and $\lambda_N(G \setminus F) < \varepsilon$.*

Proof. Fix $\varepsilon > 0$. Write $E = \bigcup_1^\infty E_n$ where each E_n is measurable, $\lambda_N(E_n) < \infty$. For each n, take open G_n such that $E_n \subseteq G_n$ and $\lambda_N(G_n) < \lambda_N(E_n) + \varepsilon/2^{n+1}$ (possible by the above Lemma). Let $G = \bigcup_1^\infty G_n \supseteq E$, G open.

$$\lambda_N(G \setminus E) \le \sum_{n=1}^{\infty} \lambda_N(G_n \setminus E_n) = \sum_{n=1}^{\infty} (\lambda_N(G_n) - \lambda_N(E_n)) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2}.$$

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By the same reasoning, there exists an open set $H \supseteq X \setminus E$ such that $\lambda_N(H \setminus (X \setminus E)) < \varepsilon/2$. Let $F = X \setminus H$ closed, $F \subseteq E$, $\lambda_N(E \setminus F) < \varepsilon/2$.

$$\lambda_N(G \setminus F) = \lambda_N(G \setminus E) + \lambda_N(E \setminus F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Corollary 1.6. *If $E \subseteq \mathbb{R}^N$ is measurable, there exists a G_δ set A and a F_σ set B such that $B \subseteq E \subseteq A$ and $\lambda_N(A \setminus B) = 0$.*

Proposition 1.7 (5.2)**.** *Let $E \subseteq \mathbb{R}^N$ be measurable, $\lambda_N(E) < \infty$, $\varepsilon > 0$. Then there exists a compact $K \subseteq E$ such that $\lambda_N(K) > \lambda_N(E) - \varepsilon$.*

Proof. Immediate from Prop 5.1 as E is bounded. If E is unbounded, there exists a bounded, measurable $E' \subseteq E$ such that $\lambda_N(E') > \lambda_N(E) - \varepsilon/2$. Unbounded case is thereby reduced to the bounded case. □

2. Lebesgue-Stieltjes Measure

Definition 2.1 (Lebesgue-Stieltjes Measure)**.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a nondecreasing, continuous function from the left. For $[a, b)$, a cell in $\mathbb{R}$, define $\mu([a, b)) = \varphi(b) - \varphi(a)$. This is a measure on the semiring of cells. Apply the extension procedure.

Definition 2.2 (Dirac Measure)**.**

$$\varphi(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

and

$$\mu_\varphi(E) = \begin{cases} 0, & 0 \notin E \\ 1, & 0 \in E \end{cases}$$

Under this measure, all $E \subseteq \mathbb{R}$ are measurable.