MATH 202A — LECTURE NOTES FOR OCT 26, 2005

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1. The Extension Theorem

Definition 1.1 (σ-finite)**.** A measure on a ring R is σ-finite if every set in R is a countable union of sets of finite measure.

Definition 1.2. For $\mathcal{F}$ a family of sets,

- $\mathcal{F}_\sigma$ is the family of countable unions of sets in $\mathcal{F}$.
- $\mathcal{F}_\delta$ is the family of countable intersections of sets in $\mathcal{F}$.

Theorem 1.3. *Let R be a ring, μ^* its outer measure corresponding to μ on the hereditary σ-ring $\mathcal{H}$ generated by R, $\mathcal{M}$ the σ-ring of μ^*-measurable sets. Then*

1. *$R \subseteq \mathcal{M}$, and $\mu^*|_R = \mu$.*
2. *If μ is σ-finite, then $\mathcal{M}$ is the σ-ring generated by R and the μ^*-null sets.*

Proof. We present this proof in two parts.

1. Let $A \in R$. We want to show $\mu^*(A) = \mu(A)$. It is clear that $\mu^*(A) \le \mu(A)$, so we show the other direction.

   Suppose $A_1, A_2, \dots \in R$ and $A \subseteq \bigcup_1^\infty A_n$. Let $B_1 = A_1$, and $B_n = A_n \setminus (A_1 \cup \dots \cup A_{n-1})$ for $n > 1$, and $A \subseteq \bigcup_1^\infty B_n$ a disjoint union. Thus

   $$\mu(A) \le \sum_{n=1}^{\infty} \mu(B_n) \le \sum_{n=1}^{\infty} \mu(A_n)$$

   Hence $\mu(A) \le \mu^*(A)$.

   Now we want to show that $A \in \mathcal{M}$. First take $S \in \mathcal{H}$. We want to show $\mu^*(S \cap A) + \mu^*(S \setminus A) \le \mu^*(S)$. This is obvious if $\mu^*(S) = \infty$, so we assume $\mu^*(S) < \infty$.

   Fix $\varepsilon > 0$. Take $A_1, A_2, \dots \in R$ such that $S \subseteq \bigcup_1^\infty A_n$ and $\sum_1^\infty \mu(A_n) < \mu^*(S) + \varepsilon$.

   $$\begin{aligned} \mu^*(S \cap A) + \mu^*(S \setminus A) &\le \sum_{n=1}^{\infty} \mu(A_n \cap A) + \sum_{n=1}^{\infty} \mu(A_n \setminus A) \\ &= \sum_{n=1}^{\infty} \mu(A_n) \\ &< \mu^*(S) + \varepsilon \\ &= \mu^*(S \cap A) + \mu^*(S \setminus A) \text{ since } \varepsilon \text{ is arbitrary} \\ &\le \mu^*(S) \end{aligned}$$

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(2) Let $E \in \mathcal{M}$, $\mu(E) < \infty$. Let $\varepsilon > 0$. Then we want to show there exists $G \in R_\sigma$ and $F \in R_\delta$ such that $F \subseteq E \subseteq G$ and $\mu(G \setminus F) < \varepsilon$.

Let $A_1, A_2, \cdots \in R$ such that $E \subseteq \bigcup_1^\infty A_n$ and $\sigma_1^\infty \mu(A_n) < \mu(E) + \varepsilon/3$. Let $G = \bigcup_1^\infty A_n = R_\sigma$, $E \subseteq G$, and $\mu(G) < \mu(E) + \varepsilon/3$. Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \overset{(m\to\infty)}{\longrightarrow} \mu(G)$$

Take m such that $\mu(\bigcup_1^m A_n) > \mu(G) - \varepsilon/3$. Let $A = \bigcup_1^m A_n \in R$ and $\mu(G \setminus E) = \mu(G) - \mu(E) < \varepsilon/3$. Take $B_1, B_2, \cdots \in R$ such that $G \setminus E \subseteq \bigcup_1^\infty B_n$ and $\sum_1^\infty \mu(B_n) < \varepsilon/3$. Let $F = A \setminus \bigcup_1^\infty B_n = \bigcap_1^\infty (A \setminus B_n) \in R_\delta$. Then

$$F \subseteq G \setminus \bigcup_{n=1}^{\infty} B_n \subseteq G \setminus (G \setminus E) = E.$$

But

$$E \setminus F \subseteq G \setminus F \subseteq (G \setminus A) \cup (A \setminus F) \subseteq (G \setminus A) \cup \left(\bigcup_{n=1}^{\infty} B_n\right),$$

So then

$$\begin{aligned}
\mu(E \setminus F) &\leq \mu(G \setminus A) + \mu\left(\bigcup_{n=1}^{\infty} B_n\right) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} \\
\mu(G \setminus F) &= \mu(G \setminus E) + \mu(E \setminus F) \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

Next we want to show that if $E \in \mathcal{M}$, then there exists $G \in (R_\sigma)_\delta$, $F \in (R_\delta)_\sigma$ such that $F \subseteq E \subseteq G$ and $\mu(G \setminus F) = 0$.

Since μ is σ-finite on R, it is σ-finite on $\mathcal{M}$. Thus $E = \bigcup_{j=1}^\infty E_j$, with $E_j \in \mathcal{M}$, $\mu(E_j) < \infty$. By the above, for each $n \in \mathbb{N}$ and each j, there exists $G_{jn} \in R_sigma$ such that $E_j \subseteq G_{jn}$ and $\mu(G_{jn} \setminus Ej) < 2^{-j}/n$. Let $G_n = \bigcup_{j=1}^\infty G_{nj} \in R_\sigma$, $E \subseteq G_n$, and $G_n \setminus E \subseteq \bigcup_{j=1}^\infty (G_{nj} \setminus E_j$. Therefore

$$\mu\left(G_n \setminus E\right) < \sum_{j=1}^{\infty} \frac{2^{-j}}{n} = \frac{1}{n}.$$

Let $G = \bigcap_1^\infty G_n \in (R_\sigma)_\delta$ and $\mu(G \setminus E) = 0$ $(E \subseteq G)$.

For each j, each $n \in \mathbb{N}$, there exists by the above a set $F_{jn} \in R_\delta$ such that $F_{jn} \subseteq E_j$ and $\mu(E_j \setminus E_{jn}) < 1/n$. Let $F_j = \bigcup_{n=1}^\infty F_{jn} \in (R_\delta)_\sigma$, $F_j \subseteq E_j$, and $\mu(E_j \setminus F_j) = 0$. Let $F = \bigcup_1^\infty F_j \in (R_\delta)_\sigma$ and $F \subseteq E$. Also,

$$E \setminus F = \left(\bigcup_{j=1}^{\infty} E_j\right) \setminus \left(\bigcup_{j=1}^{\infty} F_j\right) \subseteq \bigcup_{j=1}^{\infty} (E_j \setminus F_j)$$

Thus $\mu(E \setminus F) = 0$.

□