

MATH 202A — LECTURE NOTES FOR OCT 19, 2005

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1. SEMIRINGS

Definition 1.1 (Semiring). A semiring (Folland: an *elementary family*) on X is a family $\mathcal{S} \subseteq \mathcal{P}(X)$ such that

- (1) $\emptyset \in \mathcal{S}$
- (2) If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$
- (3) If $A, B \in \mathcal{S}$, then $A \setminus B$ is a finite disjoint union of sets in \mathcal{S}

Example 1.2. A cell in \mathbb{R} is $[a, b)$ or \emptyset . The cells in \mathbb{R} form a semiring.

Example 1.3. A cell in \mathbb{R}^N ($N > 1$) is \emptyset or a product of cells $\prod_{i=1}^N [a_i, b_i)$. Cells in \mathbb{R}^N form a semiring.

Proposition 1.4. Let \mathcal{S} be a semiring of X and let \mathcal{R} be the family of finite disjoint unions of sets in \mathcal{S} . Then \mathcal{R} is the ring generated by \mathcal{S} (i.e. in particular \mathcal{R} is a ring).

2. MEASURE

Definition 2.1 (Measure). Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a semiring. A measure on \mathcal{S} is a function $\mu : \mathcal{S} \rightarrow [0, \infty]$ (if μ maps to $[0, \infty)$, we say μ is a finite measure) such that

- (1) $\mu(\emptyset) = 0$
- (2) If $A_1, A_2, \dots \in \mathcal{S}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Example 2.2. Let \mathcal{S} be a semiring of cells in \mathbb{R}^N . For $C = \prod_{i=1}^N [a_i, b_i) \in \mathcal{S}$. Let $\mu(C) = \prod_{i=1}^N (b_i - a_i)$ and $\mu(\emptyset) = 0$. This is a measure, but it's nontrivial to prove it ("proof by handout").

Example 2.3. Let \mathcal{S} be the semiring of cells in \mathbb{R} . Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be nondecreasing, continuous from the left. For $C = [a, b)$, define $\mu(C) = \varphi(b) - \varphi(a)$, $\mu(\emptyset) = 0$. This is also a measure.

Example 2.4. Let $\rho : X \rightarrow [0, \infty]$, $\mathcal{S} \subseteq \mathcal{P}(X)$. Let $\mu(A) = \sum_{x \in A} \rho(x)$. If $\rho : X \rightarrow 1$, then $\mu(A) = |A|$ (the "counting measure").

Theorem 2.5 (Baby Extension Theorem). Let μ be a measure on a semiring \mathcal{S} and let \mathcal{R} be the ring generated by \mathcal{S} . Then μ has a unique extension to a measure on \mathcal{R} .

The tricky part about the proof of this theorem is that a set in \mathcal{R} can be represented in many different ways.

Lemma 2.6. *Let \mathcal{S} and μ be as above. Let $A_1, A_2, \dots, \in \mathcal{S}$ be pairwise disjoint, $B_1, B_2, \dots \in \mathcal{S}$ and assume $\bigcup A_n \subseteq \bigcup B_n$. Then $\sum \mu(A_n) \leq \sum \mu(B_n)$.*

Proposition 2.7. *Let μ be a measure on a ring \mathcal{R} , and $E_1 \subseteq E_2 \subseteq \dots \in \mathcal{R}$. Then $E = \bigcup_1^\infty E_n \in \mathcal{R}$, then $\mu(E) = \lim \mu(E_n)$.*

Proposition 2.8. *Let μ, \mathcal{R} be as above and $E_1 \supseteq E_2 \supseteq \dots \in \mathcal{R}$, $\mu(E_1) < \infty$. Let $E = \bigcap_1^\infty E_n \in \mathcal{R}$ then $\lim \mu(E_n) = \mu(E)$.*

2.1. Outer Measures.

Definition 2.9 (Hereditary). A family $\mathcal{H} \subseteq \mathcal{P}(X)$ is hereditary if $A \subseteq \mathcal{H}$, $B \subseteq A$, then $B \in \mathcal{H}$.

Definition 2.10 (Outer Measure). Let $\mathcal{H} \subseteq \mathcal{P}(X)$ be a hereditary σ -ring. An outer measure on \mathcal{H} is a function $\mu^* : \mathcal{H} \rightarrow [0, \infty)$ such that

- (1) $\mu^*(\emptyset) = 0$
- (2) If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$
- (3) If $A_1, A_2, \dots \in \mathcal{H}$ then

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

Remark 2.11. Suppose μ is a measure on a ring \mathcal{R} on X . Let \mathcal{H} be the family of subsets of X coverable by countable subfamilies of \mathcal{R} .

Define $\mu^* : \mathcal{H} \rightarrow [0, \infty)$ by

$$\mu^*(S) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) \mid A_1, A_2, \dots \in \mathcal{R}, S \subseteq \bigcup_{n=1}^{\infty} A_n \right\}.$$

We assert that μ^* is an outer measure.

Definition 2.12 (μ -measurable). If μ^* is an outer measure on \mathcal{H} , then we say a set $E \in \mathcal{H}$ is μ -measurable if $\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$ for all $S \in \mathcal{H}$.

Remark 2.13. It is always true that

$$\mu^*(S) \leq \mu^*(S \cap E) + \mu^*(S \setminus E)$$

To prove that E is μ^* -measurable, only need to prove

$$\mu^*(S \cap E) + \mu^*(S \setminus E) \leq \mu^*(S), \forall S \in \mathcal{H}$$

Definition 2.14 (μ^* -null). $E \in \mathcal{H}$ is μ^* -null if $\mu^*(E) = 0$.

Proposition 2.15. μ^* -null sets are μ^* -measurable.

Theorem 2.16 (3.1). *Let μ^*, \mathcal{H} be as above. Let \mathcal{M} be the family of μ^* -measurable sets. Then \mathcal{M} is a σ -ring and $\mu^*|_{\mathcal{M}}$ is a measure.*