# MATH 202A — LECTURE NOTES FOR OCT 17, 2005

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## 1. Compactifications

**Definition 1.1** (Stone-Cech Compactification)**.** $X$ Tychonoff space, noncompact, $\Phi$ a family of continuous $\varphi : X \to [0,1]$, and $X \xrightarrow{e} [0,1]^\Phi$ an embedding $\beta X = \overline{e(X)}$. Then we call $(\beta X, e)$ the Stone-Cech compactification of $X$.

**Theorem 1.2.** *Let $Z$ be compact, $T_2$, and $f : X \to Z$ continuous. Then there exists a continuous function $g : \beta X \to Z$ such that $f = g \circ e$.*

*Proof.* In the simple case, we let $Z = [0,1]$, i.e. $f \in \Phi$. Then $f = \pi_f \circ e$. The function $g = \pi_f|_{\beta X}$ does the job.

In the general case, we do some fancy commutative diagram stuff.

In the case $X = \mathbb{N}$, any bounded sequence on $\mathbb{N}$ is the restriction of a continuous function on $\beta\mathbb{N}$. The cardinality of $\beta\mathbb{N} \setminus \mathbb{N}$ is very big. It is $2^{2^{\aleph_0}}$. In addition, $\beta\mathbb{N}$ is what is called extremely disconnected: the closure of every open set is clopen. Also, $\beta\mathbb{N}$ is compact, but not sequence compact (consider $\mathbb{N}$). □

## 2. Measure and Integration

We consider some nonempty set $X$, which may or may not have a topology.

**Definition 2.1** (Ring)**.** A ring of sets on $X$ is a family of sets that contains $\emptyset$, is closed under finite unions, and is closed under relative complementation.

**Definition 2.2** ($\sigma$-Ring)**.** A $\sigma$-ring is a ring that is closed under countable unions.

**Definition 2.3** (Algebra)**.** An algebra is a ring containing $X$.

**Definition 2.4** ($\sigma$-Algebra)**.** A $\sigma$-algebra is a $\sigma$-ring containing $X$.

*Example* 2.5. Some examples:

(1) $\{X, \varphi\}$ is a $\sigma$-algebra.
(2) $2^X$ is a $\sigma$-algebra.
(3) If $X$ is uncountable, its countable subsets form a $\sigma$-ring, not a $\sigma$-algebra.

**Proposition 2.6.** *A ring is closed under finite intersections, a $\sigma$-ring under countable intersections.*

*Proof.* Let $\mathcal{R}$ be a ring on $X$ and let $A_1, A_2, \ldots, A_n \in \mathcal{R}$. Then

$$\bigcap_{j=1}^{n} A_j = A_1 \setminus \bigcup_{j=2}^{n} (A_1 \setminus A_j)$$

And the same proof works for the $\sigma$-ring case. □

Scribed by Chris Crutchfield Last updated October 18, 2005.

*Remark* 2.7. Some remarks:

(1) The intersection of any family of rings on $X$ is a ring.
(2) Any subset of $2^X$ is contained in a smallest ring on $X$, the ring generated by the family.
(3) Similarly, for $\sigma$-rings, algebra, $\sigma$-algebras.

**Definition 2.8** (Borel $\sigma$-Algebra)**.** If $X$ is a topological space, the $\sigma$-algebra generated by the open sets of $X$ is called the Borel $\sigma$-algebra.