

MATH 202A — LECTURE NOTES FOR OCT 12, 2005

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1. EMBEDDING, METRIZATION, COMPACTIFICATION

Definition 1.1 (Embedding). let X and Y be compact metric spaces. An embedding of X into Y is a one-to-one map $e : X \rightarrow Y$ which is a homeomorphism of X onto $e(X)$.

Definition 1.2 (Distinguishing Family). X is a topological space, Φ is a family of continuous functions $\varphi : X \rightarrow Y_\varphi$ (a topological space). We say Φ distinguishes points of X if for all $a, b \in X$, $a \neq b$, then there exists $\varphi \in \Phi$ such that $\varphi(a) \neq \varphi(b)$.

We also say Φ distinguishes points of X from closed sets if for $F \subseteq X$ closed, and $a \in X \setminus F$, there exists $\varphi \in \Phi$ such that $\varphi(a) \notin \overline{\varphi(F)}$.

Lemma 1.3 (Embedding Lemma). Let X, Φ, Y_φ be as above. Let $Y = \prod_{\varphi \in \Phi} Y_\varphi$. Define $e : X \rightarrow Y$ (the evaluation map), by $e(x) = (\varphi(x))_{\varphi \in \Phi}$ (i.e. $\pi_\varphi(e(x)) = \varphi(x)$). Then

- (1) e is continuous
- (2) e is one-to-one if and only if Φ distinguishes points of X
- (3) If Φ distinguishes points of X from closed sets, then e is an open map of X onto $e(X)$.

Proof. (1) and (2) are obvious. We want to show a proof for (3).

Assume Φ distinguishes points from closed sets. Take $U \subseteq X$, $a \in U$. We want to show there exists a neighborhood W of $e(a)$ in Y such that $W \cap e(X) \subseteq e(U)$. We take $\psi \in \Phi$ such that $\psi(a) \notin \overline{\psi(X \setminus U)}$, i.e. $\psi(a) \in Y_\psi \setminus \overline{\psi(X \setminus U)}$ (which is open in Y_ψ). Then $e(a) \in \pi_\psi^{-1}(V) \cap e(X)$. Consider $x \in X$ such that $\psi(x) = \pi_\psi(e(x)) \in V$. Then $\psi(x) \notin \overline{\psi(X \setminus U)}$, so $x \in U$. Hence $e(x) \in \pi_\psi^{-1}(V)$, which implies $e(x) \in e(U)$. We have $e(a) \in \pi_\psi^{-1}(V) \cap e(X) \subseteq e(U)$. Take $W = \pi_\psi^{-1}(V)$. \square

Corollary 1.4. If Φ distinguishes points of X and distinguishes points from closed sets, then e is an embedding of X into Y .

Definition 1.5 (Completely Regular). X is completely regular if for $A \subseteq X$ closed, $b \notin A$, there exists continuous $\varphi : X \rightarrow [0, 1]$ such that $\varphi(A) = 0$ and $\varphi(b) = 1$.

Definition 1.6 (Tychonoff Space). If X is completely regular and T_1 , it is called a Tychonoff space (in between T_3 and T_4).

Remark 1.7. We see the following:

- (1) A Tychonoff space is T_2 .
- (2) A T_4 space is a Tychonoff space (by Urysohn's Lemma). Hence a compact T_2 space is a Tychonoff space.

(3) Any subspace of a Tychonoff space is a Tychonoff space.

Definition 1.8 (Cube). A cube is a product of copies of $[0, 1]$. $[0, 1]^{\mathbb{N}}$ is called the Hilbert cube.

Theorem 1.9. X is a Tychonoff space if and only if it can be embedded in a cube.

Proof. A cube is a compact T_2 space, hence Tychonoff. Therefore its subspaces are Tychonoff. Now we show the other direction.

Let X be Tychonoff. Let Φ be the family of continuous functions of X into $[0, 1]$. Then Φ distinguishes points of X and distinguishes points from closed sets. By the embedding lemma, the evaluation map $e : X \rightarrow [0, 1]^{\Phi}$ is an embedding. \square

Definition 1.10 (Metrizability). X is metrizable if there exists a metric on X that induces its topology.

Example 1.11. The Hilbert cube, $[0, 1]^{\mathbb{N}}$ is metrizable by the metric

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|.$$

Theorem 1.12 (Urysohn's Metrization Theorem). A second countable, regular T_1 space is metrizable.

Proof. Next time. \square

Lemma 1.13 (Tychonoff's Lemma). A second countable, regular space is normal.

Proof. Let \mathcal{B} be the countable base of open sets. Let $A, B \subseteq X$ be closed, nonempty, and disjoint. For each $a \in A$, there exists an open set containing a whose closure is disjoint from B . We can cover A by a countable family $\{U_1, U_2, \dots\}$ of open sets such that $\overline{U_n} \cap B = \emptyset$. In addition, we can cover B by a countable family $\{V_1, V_2, \dots\}$ of open sets such that $\overline{V_n} \cap A = \emptyset$. Define G_1, G_2, \dots and H_1, H_2, \dots as follows:

$$G_n = U_n \setminus (\overline{V_1} \cup \dots \cup \overline{V_n})$$

$$H_n = V_n \setminus (\overline{U_1} \cup \dots \cup \overline{U_n})$$

and let $G = \bigcup_{n=1}^{\infty} G_n$, $H = \bigcup_{n=1}^{\infty} H_n$ open. Then $A \subseteq G$, $B \subseteq H$, and $G \cap H = \emptyset$. \square