MATH 202A — LECTURE NOTES FOR OCT 10, 2005

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1. More stuff

Lemma 1.1. *For $R > 0$, there exists a sequence $(p_n)_1^\infty$ of polynomials such that $p_n(t) \to |t|$ uniformly on $[-R, R]$.*

Definition 1.2 (Lattice)**.** For $f, g \in C(X)$, we let $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$. Note that $f \vee g, f \wedge g \in C(X)$. $L \subseteq C(X)$ is a lattice if it is closed under $\vee$ and $\wedge$.

Lemma 1.3. *Given the same assumptions as in Theorem 1.4, let $L \subseteq C(X)$ be a lattice. Then $h \in C(X)$ belongs to $\overline{L}$, if for each $a, b \in X$, $\varepsilon > 0$, there exists $f \in L$ such that $|h(a) - f(a)| < \varepsilon$ and $|h(b) - f(b)| < \varepsilon$.*

Proof. Assume h satisfies the conditions. For $a, b \in X$, $a \neq b$, choose $f_{a,b} \in L$ such that $|h(a) - f_{a,b}(a)| < \varepsilon$ and $|h(b) - f_{a,b}(b)| < \varepsilon$. Let $U_{a,b} = \{\, x \mid |h(x) - f_{a,b}(x)| < \varepsilon \,\}$ be a neighborhood of a and b.

We fix a. Then $\{\, U_{a,b} \mid b \in X \,\}$ is an open cover of X. We take a finite subcover $\{U_{a,b_1}, U_{a,b_2}, \ldots, U_{a,b_n}\}$ and let $f_a = f_{a,b_1} \vee \cdots \vee f_{a,b_n} \in L$. Then $U_a = U_{a,b_1} \cap \cdots \cap U_{a,b_n}$ is an open neighborhood of a.

Then $f_a(x) > h(x) - \varepsilon$ for all $x \in X$, and thus $|h(x) - f_a(x)| < \varepsilon$ for all $x \in U_a$. We note that the set of U_a form an open cover of X. Thus there exists a finite subcover $\{U_{a_1}, \ldots, U_{a_m}\}$. We let $f = f_{a_1} \wedge \cdots \wedge f_{a_m}$. Then $f \in L$ and $h(x) - \varepsilon < f(x) < h(x) + \varepsilon$ for all $x \in X$. Hence $\|h - f\|_\infty < \varepsilon$. ☐

Theorem 1.4 (Stone-Weierstrass Theorem)**.** *Let X be a compact T_2 space, $A \subseteq C(X)$ a closed subalgebra the separates points of X. Then either $A = C(X)$ or there exists $x_0 \in X$ such that $A = \{\, f \in C(X) \mid f(x_0) = 0 \,\}$.*

Proof. We will break this proof into steps.

(1) If $f \in A$, then $|f| \in A$.
Let $R = \|f\|_\infty$, and apply Lemma 1.1 to get a sequence of polynomials $(p_n)_1^\infty$ such that $p_n(t) \to |t|$ uniformly on $[-R, R]$. The functions $p_n(t)$ are in A and $p_n(t) \to |f|$ uniformly, so $|f| \in A$.

(2) A is a lattice. Let $f, g \in A$. Then

$$f \vee g = \frac{f+g}{2} + \frac{|f-g|}{2} \in A$$

$$f \wedge g = \frac{f+g}{2} - \frac{|f-g|}{2} \in A$$

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(3) If A contains a nonvanishing function then $A = C(X)$. We show that for $a, b \in X$, $a \neq b$, $\alpha, \beta \in \mathbb{R}$. Then there exists $f \in A$ such that $f(a) = \alpha$, $f(b) = \beta$. By Lemma 1.3, this will imply $A = C(X)$. Take $g_1 \in A$ such that $g_1(a) \neq g_1(b)$ (it separates a and b). Take $g_2 \in A$ such that $g_2(a) \neq 0 \neq g_2(b)$. Some linear combination g of these will satisfy both $g(a) \neq 0 \neq g(b)$ and $g(a) \neq g(b)$ (by basic linear algebra). Let $\xi = g(a)$ and $\eta = g(b)$. Then there exists $\gamma, \delta \in \mathbb{R}$ such that $f = \gamma g + \delta g^2$ satisfies $f(a) = \alpha$, $f(b) = \beta$. So $\xi\gamma + \xi^2\delta = \alpha$, $\eta\gamma + \eta^2\gamma = \beta$.

(4) If $f^{-1}(0) \neq \emptyset$ for all $f \in A$, then there exists $x_0 \in X$ such that $A = \{ f \mid f(x_0) = 0 \}$. Let $Z_f = f^{-1}(0)$, where $f \in A$, closed, $\neq \emptyset$. Then $\{ Z_f \mid f \in A \}$ has FIP. Thus if $f_1, f_2, \ldots, f_n \in A$, then

$$Z_{f_1} \cap \cdots \cap Z_{f_n} \supseteq Z_{f_1^2 + \cdots + f_n^2} \neq \emptyset$$

Therefore $\bigcap_{f \in A} Z_f \neq \emptyset$. In fact there exists $x_0 \in X$ such that $\bigcap_{f \in A} Z_f = \{x_0\}$ (this is singleton because A separates points.

Let $\tilde{A} = \{ f + \lambda \mid f \in A, \lambda \in \mathbb{R} \}$, a subalgebra that separates points.

We claim that $\tilde{A}$ is closed. Take a convergent sequence $g_n \in \tilde{A}$, so $g_n = f_n + \lambda_n$. Then $(g_n(x_0))_1^\infty = (\lambda_n)_1^\infty$ converges, say to λ. Therefore $(g_n - \lambda_n)_1^\infty = (f_n)_1^\infty$ converges uniformly, say to $f \in A$. Thus $\lim g_n = f + \lambda \in \tilde{A}$. By step 3, $\tilde{A} = C(X)$, which implies $A = \{ f \in C(X) \mid f(x_0) = 0 \}$.

$\square$