MATH 202A — LECTURE NOTES FOR OCT 7, 2005

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Theorem 1 (Tietze Extension Theorem)**.** *If X is a compact Hausdorff space with $X_0 \subseteq X$ closed and nonempty, and $f_0 : X_0 \to [\alpha, \beta]$ is continuous, then there exists a continuous function $f : X \to [\alpha, \beta]$ such that $f|_{X_0} = f_0$.*

Proof. Without loss of generality, let $[\alpha, \beta] = [-1, 1]$. Let $A = f_0^{-1}([-1, -1/3])$, $B = f_0^{-1}([1/3, 1])$ be closed in X_0. By Urysohn's Lemma, there exists $g_1 : X \to [-1/3, 1/3]$ such that $g_1 = -1/3$ on A and $g_1 = 1/3$ on B.

On X_0, $|f_0 - g_1| < 2/3$. Apply same reasoning with $(f_0 - g_1)|_{X_0}$ in place of f_0 to get a continuous $g_2 : X \to [-2/9, 2/9]$ such that $|f_0 - g_1 - g_2| \leq 4/9$ on X_0. Iterate this procedure to get a sequence $(g_n)_1^\infty$ of continuous functions, where g_n has range in $[-2^{n-1}/3^n, 2^{n-1}/3]$ such that

$$\left| f_0 - \sum_{k=1}^{n} g_k \right| \leq \left(\frac{2}{3}\right)^n \quad \text{on } X_0.$$

The series $\sum g_n$ converges uniformly on X to a continuous function f with values in $[-1, 1]$. By Equation , $f|_{X_0} = f_0$. □

Definition 2 (Banach Algebra)**.** Consider the space $C(X)$. Under the $\|\cdot\|_\infty$ norm, $C(X)$ is a Banach Algebra if

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty.$$

Theorem 3 (Stone-Weierstrass Theorem)**.** *Original Weierstrass Theorem (1885).*

1. *Polynomials are dense in $C[a, b]$ $(-\infty < a < b < \infty)$.*
2. *A continuous function f_n with period 2π can be uniformly approximated by finite linear combinations of the functions $\cos(nx)$ and $\sin(nx)$.*

Let X be be compact and Hausdorff, and let A be a closed subalgebra of $C(X)$ that separates the points of X.

1. *If A contains a nonvanishing function, then $A = C(X)$.*
2. *If every function in A has a zero, then there exists $x_0 \in X$ such that $A = \{ f \in C(X) \mid f(x_0) = 0 \}$.*

For Weierstrass 1, $X = [a, b]$, and let $A_0 \subseteq C[a, b]$ be restrictions of polynomials to $[a, b]$. Then by Stone-Weierstrass, $\overline{A_0} = C[a, b]$, i.e. A_0 is dense.

For Weierstrass 2, $X = [-\pi, \pi]/\{-\pi, \pi\} = S'$ (the circle).

Lemma 4 (Dini's Lemma)**.** *Let $(f_n)_1^\infty$ be a sequence in $C(X)$ converging pointwise and monotonically (non-decreasing or non-increasing) to $f \in C(X)$. Then $f_n \to f$ uniformly.*

Scribed by Chris Crutchfield — Last updated October 17, 2005.

Proof. Fix $\varepsilon > 0$. Let $K_n = \{ x \in X \mid |f(x) - f_n(x)| \geq \varepsilon \}$ for $n \in \{1, 2, \dots\}$. Then K_n is closed, $K_{n+1} \subseteq K_n$ for all n by monotonicity, and $\bigcap_1^\infty K_n = \emptyset$. The K_n are compact since X is, so there exists n such that $K_n = \emptyset$, i.e. $|f(x) - f_n(x)| < \varepsilon$ for all x. □

Lemma 5. *For $R > 0$, there exists a sequence $(p_n)_1^\infty$ of polynomials such that $p_n(t) \to |t|$ unformly on $[-R, R]$ and $p_n(0) = 0$ for all n.*

Proof. Without loss of generality, assume $R = 1$. Let $q(t) = 1 - |t|$ $(|t| \leq 1)$. It is enough to find a sequence of polynomials converging uniformly to q on $[-1, 1]$, and 1 at 0. Note: q takes values in $[0, 1]$ and q satisfies

$$(1 - q(t))^2 = t^2. \tag{5.1}$$

For a given $t \in [-1, 1]$, consider the equation $(1 - s)^2 = t^2$. It has two solutions $s = 1 - |t| \in [0, 1]$, $s = 1 + |t| > 1 \notin [0, 1]$. q is the unique function on $[-1, 1]$ that satisfies Equation 5.1 and takes values in $[0, 1]$. Rewrite Equation 5.1 as

$$q(t) = \frac{1}{2}(1 - t^2 + q(t)^2) \tag{5.2}$$

Define $q_0, q_1, \dots$ by

$$\begin{aligned} q_0(t) &= 1 \\ q_1(t) &= 1 - \frac{t^2}{2} \\ &\vdots \\ q_{n+1} &= \frac{1}{2}(1 - t^2 + q_n(t)^2) \\ &\vdots \end{aligned}$$

By induction, $q_n(0) = 1$ for all n, and q_n takes values in $[0, 1]$ for all n. Also, $q_n - q_{n+1} = (q_{n-1}^2 - q_n^2)/2 = (q_{n-1} - q_n)(q_{n-1} + q_n)/2$, so by induction: $q_{n+1} \leq q_n$ for all n. The sequence $(q_n)_1^\infty$ converges pointwise, say to $\tilde{q}$. The recurrence relation implies $\tilde{q}$ satisfies Equation 5.2, i.e. Equation 5.1. Since $\tilde{q}$ takes values in $[0, 1]$, we get $\tilde{q} = q$. Use Dini's Lemma to get $q_n \to q$ uniformly. □