MATH 202A — LECTURE NOTES FOR OCT 3, 2005

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1. Equicontinuity

Definition 1.1. Let X be a compact space, and (Y, σ) a complete metric space. Define $C(X, Y)$ to be the space of all continuous functions $f : X \to Y$, metrized by $\rho(f, g) = \sup \{ \sigma(f(x), g(x)) : x \in X \}$

Example 1.2. In the case where $Y = \mathbb{R}$, we write $C(X)$ for $C(X, \mathbb{R})$.

Proposition 1.3. *$C(X, Y)$ is complete.*

Proof. Let $(f_n)_1^\infty \subseteq C(X, Y)$ be Cauchy. For each $x \in X$, the sequence $(f_n(x))_1^\infty$ is Y is Cauchy, hence convergent, say to $f(x)$.

Now we must prove that f is continuous. Fix $\varepsilon > 0$ and note that for any choice of n,

$$\sigma(f(x), f_n(x)) = \lim_{m\to\infty} \sigma(f_n(x), f_m(x)) \leq \sup \{ \rho(f_m, f_n) : m \geq n \} \tag{1.1}$$

Choose n such that the right-hand side of the equation above is less than $\varepsilon/3$. Consider any $x_0 \in X$. Then there exists a neighborhood V of x_0 such that $\sigma(f_n(x), f_n(x_0)) < \varepsilon/3$ for all $x \in V$. Thus for $x \in V$,

$$\begin{aligned}\sigma(f(x), f(x_0)) &\leq \sigma(f(x), f_n(x)) + \sigma(f_n(x), f_n(x_0)) + \sigma(f_n(x_0), f(x_0)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon\end{aligned}$$

Therefore f is continuous at x_0.

Now Equation 1.1 can be rewritten

$$\rho(f, f_n) \leq \sup \{ \rho(f_m, f_n) : m \geq n \}$$

Which goes to 0 as n goes to ∞. Therefore $f_n \to f$ in the ρ-metric. □

Question 1.4. What are the compact subsets of $C(X, Y)$?

Definition 1.5 (Equicontinuity)**.** A subset $\mathcal{M} \subseteq C(X, Y)$ is equicontinuous at $x_0 \in X$ if for all $\varepsilon > 0$, there exists a neighborhood V of x_0 such that $\sigma(f(x), f(x_0)) < \varepsilon$ for all $x \in V$ and for all $f \in \mathcal{M}$. If this holds for all $x \in X$, we say $\mathcal{M}$ is equicontinuous.

Example 1.6. Consider $C[0, 1]$.

1. The set $\{ f \in C[0, 1] : f \text{ is differentiable and } |f'| \leq 1 \}$ is equicontinuous.
2. $\{ f \in C[0, 1] : |f(x) - f(y)| \leq |x - y| \} \subseteq Lip_1[0, 1]$
3. For $0 < \alpha \leq 1$, $\{ f \in C[0, 1] : |f(x) - f(y)| \leq |x - y|^\alpha \} \subseteq Lip_\alpha[0, 1]$

Proposition 1.7. *If $\mathcal{M}$ is totally bounded, then $\mathcal{M}$ is equicontinuous.*

Scribed by Chris Crutchfield Last updated October 17, 2005.

Proof. Fix $\varepsilon > 0$. Take $f_1, f_2, \dots, f_n \in \mathcal{M}$ such that each $f \in \mathcal{M}$ is within ρ-distance $\varepsilon/3$ of at least one of these. Let $x_0 \in X$. Take a neighborhood V of x_0 such that $\sigma(f_k(x), f_k(x_0)) < \varepsilon/3$ for all $x \in V$, $k \in \{1, 2, \dots, n\}$. Fix $f \in \mathcal{M}$, and take f_k such that $\rho(f, f_k) < \varepsilon/3$.

For $x \in V$, $\sigma(f(x), f(x_0)) \leq \sigma(f(x), f_k(x)) + \sigma(f_k(x), f_k(x_0)) + \sigma(f_k(x_0), f(x_0)) < \varepsilon$. Therefore $\mathcal{M}$ is equicontinuous at x_0. □

Theorem 1.8 (Arzelà-Ascoli Theorem)**.** *$\mathcal{M} \subseteq C(X, Y)$ is relatively compact if and only if*

(1) *$\mathcal{M}$ is equicontinuous.*
(2) *For each $x \in X$, the set $\{ f(x) : f \in \mathcal{M} \}$ is relatively compact.*

Proof. Assume $\mathcal{M}$ is relatively compact. Then $\mathcal{M}$ is totally bounded, hence equicontinuous by Proposition 1.7. For $x \in X$, the map $\varphi : C(X, Y) \to Y$, $f \mapsto f(x)$ is continuous. Hence, since $\overline{\mathcal{M}}$ is compact, its image under this map is compact. Thus $\{ f(x) : f \in \mathcal{M} \}$ is relatively compact.

Now assume that (1) and (2) hold. We want to show $\mathcal{M}$ is totally bounded. Fix $\varepsilon > 0$. For each $x \in X$, there exists a neighborhood V_x of x such that $\sigma(f(x), f(x')) < \varepsilon/4$ for all $x' \in V_x$, and for all $f \in \mathcal{M}$. Since X is compact, there exists a finite set $X_0 \subseteq X$ such that $\bigcup_{x \in X} V_x = X$.

For each $x \in X_0$, the set $\{ f(x) : f \in \mathcal{M} \}$ is relatively compact, hence totally bounded. Hence there exists a finite subset Y_x such that for every $f \in \mathcal{M}$, $f(x)$ is within $\varepsilon/4$ of some point of Y_x.

Finish the proof yourself. □