MATH 202A — LECTURE NOTES FOR SEPT 28, 2005

PROFESSOR DONALD SARASON

Correction

A subset of a complete metric space is relatively compact if and only if it is totally bounded.

Proposition 0.1. *A product space $\prod_{i \in \mathcal{S}} X_i$ is T_2 if and only if each X_i is T_2.*

1. Tychonoff

Theorem 1.1 (Tychonoff Product Theorem)**.** *A product of compact spaces is compact.*

Zorn's lemma will be used in the proof. But first, some definitions.

Definition 1.2 (Partially Ordered Set)**.** A partially ordered set is a set $P \neq \emptyset$ equipped with a binary relation $\leq$ such that

(1) $a \leq a$ for all a (reflexivity)
(2) $a \leq b$ and $b \leq a$ implies $a = b$ (anti-symmetry)
(3) $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity)

Definition 1.3 (Linearly Ordered Set)**.** P is linearly ordered if we have the additional condition that if $a, b \in P$, then either $a \leq b$ or $b \leq a$.

Definition 1.4 (Well-ordered)**.** A linearly ordered set is well-ordered if every nonempty subset contains a least element.

Definition 1.5 (Chain)**.** A chain in P is a linearly ordered subset of P.

Definition 1.6 (Inductivity)**.** P is called inductive if every chain in P has a least upper bound in P.

Definition 1.7 (Maximality)**.** An element of P is maximal if $b \in P$ and $b \geq a$ implies $b = a$ (i.e. there does not exist $b \in P$ such that $a < b$).

Lemma 1.8 (Zorn's Lemma)**.** *An inductive partially ordered set contains a maximal element.*

Proof. Nota bene: A proof of Zorn's Lemma by the axiom of choice was provided in a supplemental handout, titled "Transfinite Nonsense". □

Lemma 1.9 (Zermelo's Lemma)**.** *Any nonempty set can be well-ordered.*

Proof. Let $S \neq \emptyset$ be a set. Let P consist of all pairs $(T, \mathcal{O})$, where $T \subseteq S$, $T \neq \emptyset$, and $\mathcal{O}$ is a well-ordering of T.

Say $(T_1, \mathcal{O}_1) \leq (T_2, \mathcal{O}_2)$ if $T_1 \subseteq T_2$ and $\mathcal{O}_2$ coincides with $\mathcal{O}_1$ on T_1. This is a partial order of P. Consider a chain C in P. Let

$$T' = \bigcup \{ T \mid (T, \mathcal{O}) \in C \} .$$

Scribed by Chris Crutchfield 	Last updated September 28, 2005.

For $a, b \in T'$, pick $(T, \mathcal{O}) \in C$ such that $a, b \in T$. Define $\mathcal{O}'$ on T' by using $\mathcal{O}$. The result is independent of $(T, \mathcal{O})$. Thus $(T', \mathcal{O}) \in P$, whish is the least upper bound of C. Hence P is inductive.

Then, by Zorn's Lemma, P contains a maximal element $(\tilde{T}, \tilde{\mathcal{O}})$. Now we want to show that $\tilde{T} = S$.

"Now, suppose it ain't!" Take $s \in S \setminus \tilde{T}$. Let $\tilde{\tilde{T}} = \tilde{T} \cup \{s\}$. Define $\tilde{\tilde{\mathcal{O}}}$ on $\tilde{\tilde{T}}$ by letting it agree with $\tilde{\mathcal{O}}$ on $\tilde{T}$ and making s strictly greater than every element of $\tilde{T}$. Then $(\tilde{\tilde{T}}, \tilde{\tilde{\mathcal{O}}}) \in P$ and $(\tilde{T}, \tilde{\mathcal{O}}) < (\tilde{\tilde{T}}, \tilde{\tilde{\mathcal{O}}})$, contrary to the maximality of $(\tilde{T}, \tilde{\mathcal{O}})$. Therefore $\tilde{T} = S$. □

Remark 1.10. There is not enough time remaining to prove the Tychonoff Product Theorem in full, so we will just demonstrate the case of countably many compact metric spaces.

Proof. Let (X_n, ρ_n) for $n \in \mathbb{N}$ be compact metric spaces. Let

$$X = \prod_{n \in \mathbb{N}} X_n$$

Each ρ_n is bounded, so without loss of generality let $\rho_n(x, y) \leq 1$ for all n and for all $x, y \in X_n$.

Define $\rho : X \times X \to [0, \infty)$ by $\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \rho_n(x_n, y_n)$. Then ρ induces the product topology. We take a sequence $(x_m)_{m=1}^{\infty}$ in X, where $x_m = (x_{m,n})_{n=1}^{\infty}$. We want to prove there exists a convergent subsequence.

Consider the sequence $(x_{m,1})_1^{\infty} \subseteq X_1$. It has a convergent subsequence $(x_{m_k,1})_{k=1}^{\infty}$. Let $x'_k = x_{m_k}$ for $k \in \mathbb{N}$. Apply the same reasoning to get a subsequence (x''_k) of (x'_k) whose second coordinates converge. Continue in this way to get a sequence $x^j = (x^j_k)_{k=1}^{\infty}$ of subsequences suc that x^{j+1} is a subsequence of x^j and the j^{th} coordinates of x^j converge. Define $x \in \prod_{n \in \mathbb{N}} X_n$ by $x(n) = x^n_n$. Then x is a...

At this point class ended and Professor Sarason told us to try to finish the proof.

□