MATH 202A — LECTURE NOTES FOR SEPT 26, 2005

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1. Compactness

Definition 1.1 (Relatively Compact/Precompact)**.** $S \subseteq X$ is relatively compact (or precompact) if $\overline{S}$ is compact.

Remark 1.2. In a metric space, relative compactness implies total boundedness. (This is a corollary of the theorem from last time).

Definition 1.3 (Local Compactness)**.** X is locally compact if each point of X has a neighborhood base consisting of compact sets.

Example 1.4. $\mathbb{R}^n$ is locally compact.

Facts

(1) A finite dimensional normed vector space is homeomorphic to $\mathbb{R}^n$ (Any norm!), hence locally compact. Not true for infinite dimensional normed vector spaces.
(2) An infinite dimensional normed vector space is not locally compact.

Example 1.5. In $\mathcal{C}_0$, the sequences e_n defined by $e_n(k) = 1$ if $k = n$ and 0 if $k \neq n$ satisfy $\|e_n\|_\infty = 1$, $\|e_n - e_m\| = 1$ if $m \neq n$. This is a bounded sequence with no convergent subsequence. Therefore $\overline{B_1(0)}$ is not compact.

Definition 1.6 (Product Space)**.** Let $(X_i)_{i\in\mathcal{I}}$ be an indexed family of topological spaces. The product space $X = \prod_{i\in\mathcal{I}} X_i$ is the set of maps from $\mathcal{I} \to \bigcup_{i\in\mathcal{I}} X_i$:

$$X = \{\, x : i \mapsto x_i \mid x_i \in X_i, \forall i \in \mathcal{I} \,\}$$

Note that X is nonempty by the axiom of choice. If $X_i = X_0$ for all i, we write $X = X_0^{\mathcal{I}}$.

For $i \in \mathcal{I}$ define $\pi_i : X \to X_i$ by $\pi_i(x) = x_i$, the projection map. For $\mathcal{J} \subseteq \mathcal{I}$, with $\mathcal{J}$ nonempty, define $\pi_{\mathcal{J}} : X \to \prod_{j\in\mathcal{J}} X_j$ by $\pi_{\mathcal{J}}(x) = x|_{\mathcal{J}}$.

Definition 1.7 (Product Topology)**.** We make $X = \prod_{i\in\mathcal{I}} X_i$ into a topological space by using the topology having as a subbase

$$\left\{\, \pi_i^{-1}(U_i) \mid i \in \mathcal{I}, U_i \subseteq X_i, U_i \text{ open} \,\right\}.$$

The corresponding base consists of sets in the form

$$\bigcap_{j\in\mathcal{J}} \pi_j^{-1}(U_j),$$

where $\mathcal{J} \subseteq \mathcal{I}$ is infinite, nonempty, and $U_j \subseteq X_j$ open. Alternatively, $U = \prod_{i\in\mathcal{I}} U_i$, $U_i \in X_i$ nonempty, open, $U_i = X_i$ except perhaps for $j \in \mathcal{J}$.

Scribed by Kevin Lin

Last updated September 27, 2005.

The product topology makes each π_i continuous, and it is the weakest such topology on X.[1]

Remark 1.8. Each π_i is an open map. (Just check for basic open sets.)

Example 1.9. $\mathbb{R}^n = \mathbb{R}^{\{1,2,3,\ldots,n\}}$. The product topology is the usual topology.

Example 1.10. Let (X_n, ρ_n) be metric spaces. The product topology on $\prod_{n\in\mathbb{N}} X_n$ is induced by the metric $\rho(x,y) = \sum_{n=1}^{\infty} \frac{2^{-n}\rho_n(x_n,y_n)}{1+\rho_n(x_n,y_n)}$. Proof left as exercise to the reader.

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[1] The remainder of the lecture was proofs on this stuff, which is pretty trivial.