

MATH 202A — LECTURE NOTES FOR SEPT 23, 2005

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1. ?

Remark 1.1. Recall that the following are equivalent:

- (1) X is countably compact.
- (2) Any countable family of closed subsets with FIP has a nonempty intersection.
- (3) Any sequence in X has a cluster point.

Proposition 1.2. *If X is countably compact, then every infinite subset of X has a limit point.*

Proof. Let $S \subseteq X$ be infinite. Take a countably infinite subset $\{x_1, x_2, \dots\} \subseteq S$, and consider the sequence $(x_n)_1^\infty$. It has a cluster point, say x . Let U be a neighborhood of x . Since the sequence is frequently in U , $U \cap S \supseteq U \cap \{x_1, x_2, \dots\}$ is infinite, hence so is $U \cap (S \setminus \{x\})$. Therefore x is a limit point of S . \square

Proposition 1.3. *If X is T_1 , and if every infinite subset of X has a limit point, then X is countably compact.*

Proof. Take a sequence $(x_n)_1^\infty$ in X . It suffices simply to prove that it has a cluster point. Let $S = \{x_1, x_2, \dots\}$. If S is finite, then there exists a constant subsequence, hence a cluster point.

So now consider S to be infinite. Take a limit point x of S . Let U be an open neighborhood of x . We want to show that the sequence is frequently in U . Fix $n_0 \in \mathbb{N}$. Let $V = U \setminus \{x_n \mid n \leq n_0, x_n \neq x\}$. V is open since X is T_1 . Since x is a limit point of S , the set $(V \setminus \{x\}) \cap S \neq \emptyset$, so there exists $n > n_0$ such that $x_n \in U$. Therefore $(x_n)_1^\infty$ is frequently in U . Therefore x is a cluster point of $(x_n)_1^\infty$. \square

Remark 1.4. In a complete metric space, there can exist closed and bounded non-compact sets.

Example 1.5. $X = \mathcal{C}_0$.

$$e_n(k) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

$e_n \in \mathcal{C}_0$, $\|e_n\|_\infty = 1$, $\|e_m - e_n\|_\infty = 1$ if $m \neq n$. Then $\{e_n \mid n \in \mathbb{N}\}$ is closed, bounded, but not compact.

Definition 1.6 (ε -bounded). Let (X, ρ) be a metric space. We say X is ε -bounded for $\varepsilon > 0$ if X has a finite cover by ε -ball. X is totally bounded if it is ε -bounded for all $\varepsilon > 0$.

Theorem 1.7. *For a metric space (X, ρ) , the following are equivalent:*

- (1) X is compact.
- (2) X is countably compact.
- (3) X is complete and totally bounded.
- (4) X is sequentially compact.

Proof. First we show (2) \Rightarrow completeness. Take a Cauchy sequence $(x_n)_1^\infty$ in X . By (2) it has a cluster point x . Fix $\varepsilon > 0$. Pick n_0 such that $\rho(x_m, x_n) < \varepsilon/2$ for all $m, n > n_0$. Pick $n_1 > n_0$, such that $\rho(x, x_{n_1}) < \varepsilon/2$. For $n \geq n_0$, $\rho(x, x_n) \leq \rho(x, x_{n_0}) + \rho(x_{n_0}, x_n) < \varepsilon$. Hence $\lim_{n \rightarrow \infty} x_n = x$.

Now we show (2) \Rightarrow totally bounded. Assume X is not totally bounded. Take $\varepsilon > 0$ such that X is not ε -bounded. Take any $x_1 \in X$. Then $X \setminus B_\varepsilon(x_1) \neq \emptyset$. Pick $x_2 \in X \setminus B_\varepsilon(x_1)$. Then $X \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2)) \neq \emptyset$. Continue in this fashion, creating a sequence $(x_n)_1^\infty$ such that $\rho(x_m, x_n) \geq \varepsilon$ for $m \neq n$. The sequence has no cluster points, so (2) fails.

Next we show (2) \Rightarrow (1). If X is countably compact, it is totally bounded, hence separable, hence second countable. But for a second countable space, countably compact implies compact (Lindelöf).

Next we show (2) \Rightarrow (4). Take a sequence $(x_n)_1^\infty$ in X . It has a cluster point x . Then $(x_n)_1^\infty$ has a subsequence converging to x .

Next we show (4) \Rightarrow (2). If (4) and $(x_n)_1^\infty$ is a sequence in X , then there exists a convergent subsequence, whose limit is a cluster point of $(x_n)_1^\infty$, showing (2) holds.

Now we show (3) \Rightarrow (2). Assume (3). Take $S \subseteq X$ infinite. We want to prove S has a limit point. Cover S by finitely many 1-balls. Pick one B_1 , such that $B_1 \cap S$ is infinite. Now cover B_1 by finitely many $1/2$ -balls. Pick one, B_2 , such that $S \cap B_1 \cap B_2$ is infinite. Continue in this way to get $\{B_n \mid n \in \mathbb{N}\}$ where B_n is a $1/n$ ball with $B_1 \cap \dots \cap B_n \cap S$ infinite. Let x_1, x_2, \dots be the centers. Then $(x_n)_1^\infty$ is a Cauchy sequence, hence convergent, say to x . Easy to see x is a limit point of S .

Ran out of time. □