MATH 202A — LECTURE NOTES FOR SEPT 21, 2005

PROFESSOR DONALD SARASON

1. Compactness

Properties of Compactness

(1) A closed subset of a compact space is compact.
(2) If f is a continuous function, f preserves compactness of sets.
(3) Let X be a Hausdorff space, $K \subseteq X$ a nonempty, compact set. If $y \in X \setminus K$, then there exists a disjoint open $U, V \subseteq X$ such that $K \subset U$, $y \in V$. In addition, K is a closed set. (A consequence of this is that compact T_2 spaces are regular).

   Proof. For $x \in K$, take disjoint open U_x, V_x such that $x \in U_x$, $y \in U_y$. Then $\{ U_x \mid x \in K \}$ is an open cover of K. Take a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ and let $U = U_{x_1} \cup \cdots \cup U_{x_n}$ and $V = V_{x_1} \cap \cdots \cap V_{x_n}$. □

(4) Let X be compact, Y as Hausdorff space. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

   Proof. Y is compact by (2). Let $g = f^{-1}$. If $F \subseteq X$ is closed, then F is compact, therefore $f(F)$ is compact, and hence closed. Thus $F \subseteq X$ is closed, which implies $g^{-1}(F)$ is closed. Therefore g is continuous. □

(5) Let X be Hausdorff, and A, B disjoint nonempty compact subsets of X. Then there exist open $U, V \subseteq X$ disjoint such that $A \subseteq U$, $B \subseteq V$. (A consequence of this is that compact T_2 spaces are normal).

   Proof. By (3), for each $y \in B$ there exists disjoint open U_y, V_y such that $A \subseteq U_y$, $y \in V_y$. Then $\{ V_y \mid y \in B \}$ is an open cover of B. Take a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Let $U = \bigcap U_{y_i}$, $V = \bigcup V_{y_i}$. □

Definition 1.1 (Finite Intersection Property)**.** A family $\mathcal{F}$ of sets has the finite intersection property (FIP) if every finite subfamily has a nonempty intersection.

Theorem 1.2. *For a topological space X, the following are equivalent.*

(1) *X is compact.*
(2) *Any family of closed subsets of X with FIP has a nonempty intersection.*
(3) *Every net in X has a cluster point.*
(4) *Every net in X has a convergent subnet.*

Proof. (3) ⇔ (4) we already know.

To show (1) ⇔ (2), we first let $\mathcal{F}$ be a family of closed subsets of X. Then let $\mathcal{U} = \{ X \setminus F \mid F \in \mathcal{F} \}$. So $\bigcap_{F \in \mathcal{F}} F = X \setminus \bigcup_{U \in \mathcal{U}} U$. So $\mathcal{U}$ covers X if and only if $\bigcap_{F \in \mathcal{F}} F = \emptyset$. The rest is up to you (only you can prevent forest fires).

Scribed by Chris Crutchfield   Last updated September 22, 2005.

We want to show (2) $\Rightarrow$ (3). We assume X is compact. Let $\xi = (\xi_\alpha)_{\alpha\in\mathcal{A}}$ be a net in X. Also, let $A_\alpha = \{\, \xi_\beta \mid \beta \succ \alpha \,\}$. Then we have this fact: $\bigcap_{\alpha\in\mathcal{A}} \overline{A_\alpha}$ is the cluster set of ξ.

Now, to show (3) $\Rightarrow$ (2). Assume (2) fails. Take a family $\mathcal{F}$ of closed subsets with FIP and empty intersection. Direct $\mathcal{F}$ by inclusion. For each $F \in \mathcal{F}$ choose $\xi_F \in F$. Any cluster point of the net $\xi = (\xi_F)_{F\in\mathcal{F}}$ must be in $\bigcap_{F\in\mathcal{F}} F$ (see above fact). Therefore the cluster set of ξ is empty. □

Definition 1.3 (Countably Compact)**.** A set X is countably compact if every countable open cover of X has a finite subcover.

Remark 1.4. If X is second countable and countably compact, then X is compact (Lindelöf's Theorem).

Theorem 1.5. *For a topological space X, the following are equivalent:*

1. *X is countably compact*
2. *Every countable family of closed subsets with FIP has a nonempty intersection.*

Definition 1.6 (Sequentially Compact)**.** X is sequentially compact if every sequence in X has a convergent subsequence.