MATH 202A — LECTURE NOTES FOR SEPT 19, 2005

PROFESSOR DONALD SARASON

1. Nets

Recall that the net $(\eta_\beta)_{\beta\in\mathcal{B}}$ is a subnet of $(\xi_\alpha)_{\alpha\in\mathcal{A}}$ if there exists $\tau : \mathcal{B} \to \mathcal{A}$ such that $\eta_\beta = \xi_{\tau(\beta)}$ and for each $\alpha \in \mathcal{A}$ there exists $\beta_0 \in \mathcal{B}$ such that $\tau(\beta) \succ \alpha$ whenever $\beta \succ \beta_0$.

Definition 1.1 (Isotone)**.** A map $\tau : \mathcal{B} \to \mathcal{A}$ is isotone if $\beta_1 \succ \beta_0$ implies $\tau(\beta_1) \succ \tau(\beta_0)$.

Definition 1.2 (Confined)**.** A subset $\mathcal{A}_0 \subseteq \mathcal{A}$ is confined in $\mathcal{A}$ is for each $\alpha \in \mathcal{A}$ there exists $\alpha_0 \in \mathcal{A}_0$ such that $\alpha_0 \succ \alpha$.

Remark 1.3. If $\tau : \mathcal{B} \to \mathcal{A}$ is isotone and $\tau(\mathcal{B})$ is confined in $\mathcal{A}$, and if $(\xi_\alpha)_{\alpha\in\mathcal{A}}$ is an $\mathcal{A}$-net, then $(\xi_{\tau(\beta)})_{\beta\in\mathcal{B}}$ is a subnet.

Proposition 1.4. *If $\xi = (\xi_\alpha)_{\alpha\in\mathcal{A}}$ is a net in X, $x \in X$, then the following are equivalent:*

1. *x is a cluster point of ξ,*
2. *ξ has a subnet converging to x.*

Proof. The proof of (2) implies (1) is trivial. So we prove the other direction.

Assume x is a cluster point of ξ. Choose a neighborhood base $\mathcal{U}$ at x. Let $\mathcal{B} = \{ (\alpha, U) \mid \alpha \in \mathcal{A}, \xi_\alpha \in U \in \mathcal{U} \}$. We say $(\alpha_1, U_1) \succ (\alpha_2, U_2)$ if $\alpha_1 \succ \alpha_2$ and $U_1 \subseteq U_2$. This is clearly reflexive and transitive by construction.

Consider $(\alpha_1, U_1), (\alpha_2, U_2) \in \mathcal{B}$. Take $\alpha_0 \in \mathcal{A}$ such that $\alpha_0 \succ \alpha_1$ and $\alpha_0 \succ \alpha_2$ by the property of directed sets. Take $U \in \mathcal{U}$ such that $U \subseteq U_1 \cap U_2$. Since ξ is frequently in U, there exists $\alpha \succ \alpha_0$ such that $\xi_\alpha \in U$. Then $(\alpha, U) \in \mathcal{B}$, $(\alpha, U) \succ (\alpha_1, U_1), (\alpha, U) \succ (\alpha_2, U_2)$. Hence $\mathcal{B}$ is a directed set under $\succ$.

Define $\tau : \mathcal{B} \to \mathcal{A}$ by $\tau(\alpha, U) = \alpha$. Then τ is isotone and $\tau(\mathcal{B})$ is confined in $\mathcal{A}$. By Remark 1.3, $\eta = (\xi_{\tau(\beta)})_{\beta\in\mathcal{B}}$ is a subnet of ξ.

Take a neighborhood $U \in \mathcal{U}$. Take $\alpha_0 \in \mathcal{A}$ such that $\xi_{\alpha_0} \in U$. Let $\beta_0 = (\alpha_0, U) \in \mathcal{B}$. Then $\eta_\beta \in U$ for all $\beta \succ \beta_0$. Thus η converges to x. □

Proposition 1.5. *Let X, Y be topological spaces. Let $f : X \to Y$, $x \in X$. Then f is continuous at x if and only if it maps nets converging to x to nets converging to $f(x)$.*

Proof. Suppose f is continuous at x. Let $\xi = (\xi_\alpha)_{\alpha\in\mathcal{A}}$ be a net in X converging to x. Let $V \subseteq Y$ be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x, so ξ is eventually in $f^{-1}(V)$, so $f \circ \xi = (f(\xi_\alpha))_{\alpha\in\mathcal{A}}$ is eventually in V. Hence $f \circ \xi$ converges to $f(x)$.

Scribed by Chris Crutchfield — Last updated September 19, 2005.

Suppose f is not continuous at x. Take a neighborhood V of $f(x)$ such that $f(U) \setminus V \neq \emptyset$ for every neighborhood U of x. Let $\mathcal{U}$ be a neighborhood base for x, directed by inclusion. For each $U \in \mathcal{U}$, choose (using the Axiom of Choice) $\xi_U \in U$ such that $f(\xi_U) \notin V$. The net $(\xi_U)_{U \in \mathcal{U}}$ converges to x but $f \circ \xi$ does not converge to $f(x)$. □

Remark 1.6. If X is first countable, the propositions hold with sequences in place of nets.

2. Compactness

Definition 2.1 (Compactness)**.** A subset C of a topological space X is compact if every open cover of C has a finite subcover.

2.1. Consequences of Compactness.

Remark 2.2. Closed subsets of compact spaces are compact.

Proof. Let X be compact, $C \subseteq X$ closed, $\mathcal{U}$ an open cover of C. Let $\mathcal{U}' = \mathcal{U} \cup \{X \setminus C\}$, an open cover of X (since C is closed). Since X is compact, it has a finite subcover of $\{U_1, \ldots, U_k\} \subseteq \mathcal{U}'$. Then $\{U_1, \ldots, U_k\} \setminus \{X \setminus C\}$ is a finite subfamily of $\mathcal{U}$ covering C. □

Remark 2.3. If $C \subseteq X$ is compact and $f : X \to Y$ is continuous, then $f(C)$ is compact.

Proof. Let $\mathcal{V}$ be an open cover of $f(C)$. Then $\{ f^{-1}(V) \mid V \in \mathcal{V} \}$ is an open cover of C. Therefore it has a finite subcover $\{f^{-1}(V_1), \ldots, f^{-1}(V_k)\}$. Then $\{V_1, \ldots, V_k\}$ covers $f(C)$. □

Note that not all compact subsets of compact spaces are closed (the converse to the first remark is not true). For T_1 spaces, it's not necessarily true.