MATH 202A — LECTURE NOTES FOR SEPT 16, 2005

PROFESSOR DONALD SARASON

1. Generalized Convergence (Moore-Smith Convergence)

Definition 1.1 (Directed Set)**.** A directed set is a set $\mathcal{A}$ equipped with a binary relation $\succ$, with the properties

1. Reflexive: $\alpha \succ \alpha$
2. Transitive: $\alpha \succ \beta$ and $\beta \succ \gamma$ implies $\alpha \succ \gamma$
3. Given $\alpha, \beta \in \mathcal{A}$, there exists $\gamma \in \mathcal{A}$ such that $\gamma \succ \alpha$ and $\gamma \succ \beta$

Example 1.2. A couple examples:

1. Take $\mathbb{N}$ or $\mathbb{R}$, or any linearly ordered subset of them ($\succ$ corresponds to $\geq$).
2. $\mathbb{R}$, where $\alpha \succ \beta$ means $|\alpha| \geq |\beta|$.
3. Let X be a topological space, and $x \in X$. Let $\mathcal{U}$ be a neighborhood base for x, directed by inclusion. $U \succ V$ means $U \subseteq V$.
4. Let X be any infinite set, and let $\mathcal{A}$ be the set of finite subsets of X, directed by inclusion.

Definition 1.3 (Nets)**.** Let $\mathcal{A}$ be a directed set, $X \neq \emptyset$ any set. A net in X based on $\mathcal{A}$ (an $\mathcal{A}$-net) is a map $\xi : \mathcal{A} \to X$. We let ξ_α (or $\xi(\alpha)$) denote the value of ξ at α, and we write $\xi = (\xi_\alpha)_{\alpha \in \mathcal{A}} = (\xi(\alpha))_{\alpha \in \mathcal{A}}$. If $S \subseteq X$, we say ξ is eventually in X if there exists $\alpha \in \mathcal{A}$ such that $\xi_\beta \in S$ for all $\beta \succ \alpha$. We say ξ is frequently in S if for each $\alpha \in \mathcal{A}$, there exists $\beta \succ \alpha$ such that $\xi_\beta \in S$.

Definition 1.4 (Net Convergence)**.** Let X be a topological space, and ξ a net in X.

We say ξ converges to the point $x \in X$ if ξ is eventually in each neighborhood of x. We say x is a cluster point of ξ if for each neighborhood of x, ξ is frequently in it.

Proposition 1.5. *For $A \subseteq X$ and $x \in X$, the following are equivalent:*

1. *x is a limit point of A.*
2. *There exists a net in $A \setminus \{x\}$ that converges to x.*

Proof. The proof that (2) implies (1) is trivial. To prove (1) implies (2), we first assume x is a limit point of A. Then we take a neighborhood base $\mathcal{U}$ of x, which is directed by inclusion ($\succ$ means $\subseteq$). For $U \in \mathcal{U}$, we choose a point $\xi_U \in U \cap (A \setminus \{x\})$ (using the axiom of choice). The net $(\xi_U)_{U \in \mathcal{U}}$ is in $A \setminus \{x\}$ and converges to x. Therefore (2) holds. ☐

Corollary 1.6. *$x \in \overline{A}$ if and only if there exists a net in A converging to x.*

Corollary 1.7. *A is closed if and only if every convergent net in A has all its limits in A.*

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Note that in the discrete topology, every net converges to every point.

Proposition 1.8. *The following are equivalent:*

1. *X is Hausdorff.*
2. *Every convergent net has a unique limit.*

Proof. The proof that (1) implies (2) is trivial. To prove (2) implies (1), we first assume that X is not Hausdorff. Then there exist $x, y \in X$ such that all neighborhoods of x and y have non-trivial intersections. Take a neighborhood base $\mathcal{U}$ for x and a neighborhood base $\mathcal{V}$ for y. Direct $\mathcal{U} \times \mathcal{V}$ by $(U_1, V_1) \succ (U_2, V_2)$ if $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$. Then for each $(U, V) \in \mathcal{U} \times \mathcal{V}$, choose a point $\xi_{U,V} \in U \times V$. Then the net $(\xi_{U,V})_{(U,V)\in\mathcal{U}\times\mathcal{V}}$ converges to x and to y. ☐

Definition 1.9 (Subnets)**.** The net $\eta = (\eta_\beta)_{\beta\in\mathcal{B}}$ is a subnet of $\xi = (\xi_\alpha)_{\alpha\in\mathcal{A}}$ if there exists $\tau : \mathcal{B} \to \mathcal{A}$ such that:

1. $\eta_\beta = \xi_{\tau(\beta)}$ for all $\beta \in \mathcal{B}$.
2. For each $\alpha \in \mathcal{A}$, there exists $\beta_0 \in \mathcal{B}$ such that $\tau(\beta) \succ \alpha$ whenever $\beta \succ \beta_0$.

Note that if η is a subnet of ξ, and ξ converges to x, then η converges to x.