MATH 202A — LECTURE NOTES FOR SEPT 14, 2005

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1. Baire's Theorems

In $\mathcal{C}_0$, consider the sequences e_n for $n = 1, 2, 3, \ldots$:

$$e_n(k) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

Let $A = \{e_1, e_2, \ldots\}$. Note that $\|e_m - e_n\|_\infty = 1$ if $m \neq n$, and that A is closed (no limit points).

Let $x \in \mathcal{C}_0$ be given by $x(k) = -1/k$. Then $\|x - e_n\| = 1 + 1/n \to 1$ as $n \to \infty$. Therefore the distance between x and A is 1.

Theorem 1.1 (Baire Category Theorem)**.** *Let (X, ρ) be a complete metric space, and $U_1, U_2, \ldots$ are dense open subsets of X. Then $\bigcap_{n=1}^\infty U_n$ is dense in X.*

Proof. Let $V \subseteq X$ be open and nonempty. We must show that $V \cap (\bigcap U_n) \neq \emptyset$. Since $V \cap U_1 \neq \emptyset$, there exists an open ball B_1 with diameter less than $1/2$, such that $\overline{B_1} \subseteq V \cap U_1$. Since $V \cap U_2 \neq \emptyset$, there exists an open ball B_2 with diameter less than $1/4$, such that $\overline{B_2} \subseteq B_1 \cap U_2$. We continue in this way to construct a sequence of open balls $B_1, B_2, B_3, \ldots$, where the diameter of B_n is less than 2^{-n}. In addition, $\overline{B_{n+1}} \subseteq B_n \subseteq \overline{B_n} \subseteq V \cap U_1 \cap \cdots \cap U_n$. Let x_n be the center of B_n. We then get a sequence $(x_n)_1^\infty$. Therefore $\rho(x_m, x_n) < \max(2^{-m}, 2^{-n})$, so $(x_n)_1^\infty$ is Cauchy. Since X is complete, this sequence converges. Let x be its limit. Then

$$x \in \bigcap_{n=1}^\infty \overline{B_n} \subseteq V \cap \left(\bigcap_{n=1}^\infty U_n \right)$$

□

Definition 1.2 (Nowhere Dense)**.** A set A in a topological space X is nowhere dense if $\operatorname{int}(\overline{A}) = \emptyset$.

Example 1.3. The Cantor set is nowhere dense in $\mathbb{R}$. $\mathbb{Q}$ is nowhere dense in $\mathbb{R}$.

Example 1.4. The boundary of a closed set is nowhere dense.

Definition 1.5 (Meager Set)**.** A meager set is a countable union of nowhere dense sets.

Example 1.6. $\mathbb{Q}$ is meager in $\mathbb{R}$ (singletons are nowhere dense).

Definition 1.7. A residual[1] is the complement of a meager set.

Theorem 1.8 (Baire's Theorem)**.** *In a complete metric space, a residual set is dense.*

Scribed by Chris Crutchfield — Last updated September 16, 2005.

[1]Professor Pugh objects to this terminology

Definition 1.9 (Baire Space)**.** A topological space in which every residual set is dense is a Baire space.

Remark 1.10 (Baire's Terminology). A meager set is a set of first category. A non-meager set is a set of second category. Baire-ness is a topological invariant.

2. Application in $C[0,1]$

For $f \in C[0,1]$, $[a,b] \subseteq [0,1]$, and a partition $p = (t_0, t_1, \ldots, t_k)$ of $[a,b]$ (where $a = t_0 < t_1 < \cdots < t_k = b$). We define

$$\ell(f,p) = \sum_{i=1}^{k} \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}$$

$$\ell(f;a,b) = \sup_{\text{finite partitions of } [a,b]} \ell(f,p)$$

If $\ell(f;0,1) < \infty$, then the graph f is rectifiable. If $\ell(f;a,b) = \infty$ for all $[a,b] \subseteq [0,1]$, then the graph f is nowhere rectifiable.

Lemma 2.1. *Let* $[a,b] \subseteq [0,1]$, $M > 0$. *Then* $U = \{ f \in C[0,1] \mid \ell(f;a,b) > M \}$ *is an open, dense subset of* $C[0.1]$.

Proof. First we show that U is open. Let $f \in U$ and let $\delta = \ell(f;a,b) - M > 0$. Pick a partition $P = (t_0, t_1, \ldots, t_k)$ of $[a,b]$ such that $\ell(f,P) > M + \delta/2$. Pick $g \in C[a,b]$ such that $\|f - g\|_\infty < \delta/4k$.

$$|f(t_j) - f(t_{j-1})| < |g(t_j) - g(t_{j-1})| + \delta/2k$$

And a few easy estimates gives $\ell(g;a,b) > M$.

Now we show that U is dense. Select any $f \in C[a,b], f \neq u$. Let $\varepsilon > 0$ and pick $g \in C[0,1]$ such that $\|g\|_\infty < \varepsilon$, but $\ell(g;a,b) > 2M$. Then

$$\ell(f+g;a,b) \geq \ell(g;a,b) - \ell(f;a,b) > 2M - M = M$$

So $f + g \in U$ and $\|f - (f+g)\|_\infty < \varepsilon$. □

Theorem 2.2. *The set of* $f \in C[0,1]$ *such that the graph* f *is nowhere rectifiable is residual.*

Proof. Let $U(m,n) = \left\{ f \in C[a,b] \;\middle|\; \ell\left(f; \frac{m-1}{n}, \frac{m}{n}\right) > n \right\}$, where n, m run over $1, 2, \ldots$.

By Lemma 2.1,

$$\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} U(m,n),$$

which is exactly the set of f such that the graph of f is nowhere rectifiable, is residual. □