MATH 202A — LECTURE NOTES FOR SEPT 12, 2005

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1. Complete Metric Spaces (cont'd)

Example 1.1. $\mathbb{R}^n, \mathcal{C}_0, \mathcal{C}[0,1], \mathcal{L}^1, \mathcal{L}^\infty$ are complete metric spaces.

$\mathcal{C}[0,1]$ are the set of continuous real-valued functions on $[0,1]$.

Normed vector spaces under the norm $\|f\|_\infty = \sup\{\, |f(x)| \mid 0 \le x \le 1 \,\}$ are complete.

$\mathcal{L}^1$ consists of real sequences $x = (x_n)_1^\infty$ such that $\|x\| = \sum_1^\infty |x_n| < \infty$.

$\mathcal{L}^\infty$ conists of bounded real sequences such that $\|x\|_\infty = \sup\{\, |x_n| \mid n \in \mathbb{N} \,\}$.

2. Completion

Let (X, ρ) be a metric space.

Definition 2.1. The sequences $(x_n)_1^\infty$ and $(y_n)_1^\infty$ are equivalent if $\lim_{n\to\infty} \rho(x_n, y_n) = 0$.

Remark 2.2. Firstly, this is an equivalence relation. Secondly, if x is Cauchy and y is equivalent to x, then y is Cauchy. Third, a cauchy sequence is equivalent to any of its subsequences.

Lemma 2.3. *If $\xi = (x_n)_1^\infty$ and $\eta = (y_n)_1^\infty$ are Cauchy, then $\lim_{n\to\infty} \rho(x_n, y_n)$ exists and depends only on the equivalence classes of ξ and η.*

Proof. We show $(\rho(x_n, y_n))_1^\infty$ is Cauchy.

$$\begin{aligned} \rho(x_m, y_m) &\le \rho(x_m, x_n) + \rho(x_n, y_n) + \rho(y_n, y_m) \\ \rho(x_m, y_m) - \rho(x_n, y_n) &\le \rho(x_m, x_n) + \rho(y_n, y_m) \to 0 \\ \limsup_{m,n\to\infty} (\rho(x_m, y_m) - \rho(x_n, y_n)) &= 0 \end{aligned}$$

Therefore $(\rho(x_n, y_n))_1^\infty$ is Cauchy, hence convergent.

Now suppose $\xi' = (x'_n)_1^\infty$ is equivalent to ξ, then

$$\rho(x'_n, y_n) \le \rho(x'_n, x_n) + \rho(x_n, y_n)$$

Therefore $\rho(x'_n, y_n) \le \rho(x_n, y_n) \to 0$, hence they are equal. □

Definition 2.4. For ξ a Cauchy sequence, $[\xi]$ denotes its equivalence class. Let $\widetilde{X}$ be the set of all equivalence classes of Cauchy sequences.

For $\xi = (x_n)_1^\infty$ and $\eta = (y_n)_1^\infty$ Cauchy sequences, let $\widetilde{\rho}([\xi], [\eta]) = \lim_{n\to\infty} \rho(x_n, y_n)$. It is easy to show that $\widetilde{\rho}$ is a metric on $\widetilde{X}$.

For $x \in X$, we let $[x]$ be the equivalence class of the sequence $(x, x, x, \dots)$, and we let $X \to \widetilde{X}_0 = \{\, [x] \mid x \in X \,\} \subseteq \widetilde{X}$, where $x \mapsto [x]$. We note that $\widetilde{\rho}([x], [y]) = \rho(x, y)$, so this preserves distance. Thus the embedding $X \to \widetilde{X}_0$ is isometric (distance preserving).

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Remark 2.5. $\widetilde{X}_0$ is dense in $\widetilde{X}$.

Proof. If $\xi = (x_n)_1^\infty$ is a Cauchy sequence in X, then $\widetilde{\rho}([\xi],[x_n]) \to 0$. □

Theorem 2.6. *$\widetilde{X}$ is complete.*

Proof. Let $(\xi_k)_{k=1}^\infty$ be a Cauchy sequence in $\widetilde{X}$. To prove it converges, it is suffices to prove that it has a convergent subsequence.

We can assume without loss of generality that $\widetilde{\rho}([\xi_k],[\xi_l]) < 1/k$ for $l < k$. We can also assume without loss of generality that, for each k, the representative $\xi_k = (x_{k_n})_{n=1}^\infty$ of $[\xi_k]$ satisfies $\rho(x_{k_m}, x_{k_n}) < 1/k$ for all m, n.

Thus for $l > k$ and for all m, n, p,

$$\begin{aligned} \rho(x_{k_m}, x_{l_n}) &\leq \rho(x_{k_m}, x_{k_p}) + \rho(x_{k_p}, x_{l_p}), \rho(x_{l_p}, x_{l_n}) \\ &\leq \frac{2}{k} + \rho(x_{k_p}, x_{l_p}) \end{aligned}$$

Let $p \to \infty$ to get

$$\rho(x_{k_m}, x_{k_n}) \leq \frac{2}{k} + \widetilde{\rho}([\xi_k],[\xi_l]) \leq \frac{3}{k}$$

Let $\xi = (x_{n_n})_1^\infty$ be a Cauchy sequence by the above. Also, $\widetilde{\rho}([\xi],[\xi_k]) = \lim \rho(x_{n_n}, x_{k_n}) \leq 3/k$. Thus $[\xi_k] \to [\xi]$ in $\widetilde{X}$. □

Theorem 2.7. *Suppose that (Y, σ) is a complete metric space and suppose that $f : X \to Y$ is an isometry and with $f(X)$ dense in Y. Then there exists a surjective, unique isometry $g : Y \to \widetilde{X}$ such that $g(f(x)) = [x]$ for all $x \in X$.*

Proof. Define g by $g(f(x)) = [x]$ (this is clearly isometric. For $y \in Y$, take a sequence (y_n) in $f(X)$ such that y_n converges to y. Then $(g(y_n))_1^\infty$ is Cauchy, and hence convergent, and its limit depends only on Y. Define $g(y)$ to be the limit. Since it is an isometry, and since Y is complete, so is $g(Y)$. Therefore $g(Y)$ is closed in $\widetilde{X}$. Since $g(Y) \supseteq \widetilde{X}_0$, which is dense in $\widetilde{X}$, it must be that $g(Y) = \widetilde{X}$. □