MATH 202A — LECTURE NOTES FOR SEPT 9, 2005

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1. Components and Connectivity

Let X be a topological space, and $C \subseteq X$. If $a \in C$, the union of all connected subsets of C containing a is a connected set, called the component of C containing a.

Properties:

(1) The component of C containing a is the maximal connected subset of C containing a.
(2) Two components of C are identical or disjoint.
(3) Components of C are relatively closed in C.

Example 1.1. For an open, nonempty subset of $\mathbb{R}$, components are open intervals.

Definition 1.2 (Totally Disconnected). C is totally disconnected if all its components are singletons.

Example 1.3. In $\mathbb{R} : \mathbb{N}, \mathbb{Q}$ and the Cantor set are totally disconnected.

Definition 1.4 (Tennary System). Each $a \in [0, 1]$ can be expressed as

$$a = \sum_{n=1}^{\infty} \frac{d_n}{3^n}, d_n \in \{0, 1, 2\}$$

The expression is unique except when a is rational and the denominator is a power of 3, $0 < a < 1$.

Example 1.5.

$$\frac{1}{3} = \frac{2}{9} + \frac{2}{27} + \cdots$$
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Definition 1.6 (Cantor Set). C, the Cantor set, consists of all a as above, such that $d_n \in \{0, 2\}$ for all n.

Example 1.7 (Construction). Let

$$C_1 = [0, 1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right),$$

where you remove all points of $(0, 1)$ where $d_1 = 1$ necessarily.

Continue this process:

$$C_2 = C_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right)$$

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and you remove all points of C_1 where $d_2 = 1$ necessarily.

$$\vdots$$

Define

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Interesting note: Even though it is an uncountable subset of the continuum, it has zero length.

Definition 1.8 (Locally Connected)**.** X is locally connected at the point $x \in X$ if x has a neighborhood base consisting of open connected sets.

Remark 1.9. X is locally connected if it is locally connected at each of its points.

Example 1.10. Normal vector spaces are locally connected.

2. Complete Metric Spaces

Let (X, ρ) be a metric space.

Definition 2.1 (Convergence)**.** A sequence $(x_n)_1^\infty$ in X converges to $x \in X$ if

$$\lim_{n\to\infty} \rho(x, x_n) = 0$$

Definition 2.2 (Cauchy Sequence)**.** $(x_n)_1^\infty$ is a Cauchy sequence if

$$\lim_{n,m\to\infty} \rho(x_n, x_m) = 0.$$

In other words, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\rho(x_m, x_n) < \varepsilon$ whenever $m, n \geq N$.

Remark 2.3. A convergent sequence is Cauchy.

Definition 2.4 (Complete Metric Space)**.** (X, ρ) is a complete metric space if every Cauchy sequence in it converges.

Example 2.5. $\mathbb{R}, \mathbb{R}^n$ are complete metric spaces.

Remark 2.6. Closed subsets of complete metric spaces are complete.

Definition 2.7 (Banach Space)**.** A Banach Space is a complete, normed vector space.

Definition 2.8. Let $\mathcal{X}_0$ be the set of all sequences $x = (x(n))_1^\infty$ in $\mathbb{R}$ such that $\lim_{n\to\infty} x(n) = 0$. Then $\mathcal{X}_0$ is a vector space over $\mathbb{R}$, normed by

$$\|x\|_\infty = \sup \{ \, |x(n)| \mid n \in \mathbb{N} \, \}$$

Claim 2.9. *$\mathcal{X}_0$ is complete.*

Proof. Consider a Cauchy sequence $(x_n)_1^\infty$ in $\mathcal{X}_0$, where $x_n = (x_n(k))_1^\infty$. For each k, $(x_n(k))_1^\infty$ is a Cauchy sequence in $\mathbb{R}$.

The proof was not finished in class. □