

MATH 202A — LECTURE NOTES FOR SEPT 7, 2005

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1. CONNECTIVITY

Let X be a topological space.

Definition 1.1. Two subsets $A, B \subseteq X$ are **separated** if $A \neq \emptyset \neq B$, $A \cap \overline{B} = \emptyset$, $\overline{A} \cap B = \emptyset$.

Example 1.2. Let $X = \mathbb{R}$. Then $A = (-\infty, 0)$, $B = (0, \infty)$ are separated. However, $A = (-\infty, 0]$, $B = (0, \infty)$ are not.

Definition 1.3. If $C \subseteq X$, a **separation** of C is a decomposition $C = A \cup B$ where A, B are separated.

Definition 1.4. C is **connected** if it has no separation.

Remark 1.5. \emptyset is connected, and singleton sets are connected.

Remark 1.6. If X is not connected, $X = A \cup B$ with A, B separated. Then $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$, and therefore A, B are both closed and open (clopen). This is because $\overline{A} \subseteq X \setminus B = A$.

In summary, X is connected if and only if its only clopen subsets are \emptyset and X .

Remark 1.7. A subset C of X is connected if and only if its only relatively clopen subsets are \emptyset and C .

Proposition 1.8. For C to be connected, it is necessary and sufficient that whenever A, B are separated, and $C \subseteq A \cup B$, then either $C \subseteq A$ or $C \subseteq B$.

Proof. Let C be connected, A, B separated, and $C \subseteq A \cup B$. If $C \cap A \neq \emptyset \neq C \cap B$, then $C = (C \cap A) \cup (C \cap B)$ would be a separation of C , which is a contradiction. Therefore the condition holds. \square

Proposition 1.9. If \mathcal{C} is a family of connected sets and $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$, then $\bigcup_{C \in \mathcal{C}} C$ is connected. (In fact, it seems you only need pairwise nonempty intersections).

Proof. Take $x \in \bigcap_{C \in \mathcal{C}} C$. Suppose A, B are separated and $\bigcup_{C \in \mathcal{C}} C \subseteq A \cup B$. Now suppose without loss of generality, $x \in A$. By Proposition 1.8, $C \subseteq A$ for all $C \in \mathcal{C}$, which implies that $\bigcup_{C \in \mathcal{C}} C \subseteq A$. Therefore by Proposition 1.8, $\bigcup_{C \in \mathcal{C}} C$ is connected. \square

Proposition 1.10. Let C be connected and let D satisfy $C \subseteq D \subseteq \overline{C}$. Then D is connected. In particular, \overline{C} is connected.

Proof. Suppose $D \subseteq A \cup B$, A, B separated. Without loss of generality, $C \subseteq A$ by Proposition 1.8. Then $D \subseteq \overline{C} \subseteq \overline{A}$, so $D \subseteq A$ since $\overline{A} \cap B = \emptyset$. By Proposition 1.8, D is connected. \square

Proposition 1.11. *The only connected subsets of \mathbb{R} are \emptyset , singletons, and intervals.*

Proof. If $C \subseteq \mathbb{R}$ is none of the above, then there exists $a, b, c \in \mathbb{R}$, $a < b < c$, such that $a, c \in C$, $b \notin C$. Then $(C \cap (-\infty, b)) \cup (C \cap (b, \infty))$ is a separation of C .

Now consider an interval $[a, b] \subseteq \mathbb{R}$. Suppose $[a, b] \subseteq A \cup B$, with A, B separated. Without loss of generality, $a \in A$. We want to prove that $[a, b] \subseteq A$. Suppose the contrary. Let $c = \inf B \cap [a, b]$. If $c \in A$, then $c \in \overline{B}$, which is a contradiction of $A \cap \overline{B} = \emptyset$. Suppose $c \in B$. Then $c > a$ and $[a, c] \subseteq A$. Therefore $c \in \overline{A}$. But, again, this contradicts $\overline{A} \cap B = \emptyset$. Therefore $[a, b]$ is connected.

Any interval is a union of finite closed intervals with a common point, and hence is connected. \square

Proposition 1.12. *Let C be connected, and let $f : X \rightarrow Y$ be a continuous map of X into the topological space Y . Then $f(C)$ is connected.*

Proof. Consider the case $C = X$, $Y = f(X)$. If Y is not connected, then $Y = A \cup B$ where A, B are disjoint, $A, B \neq \emptyset$, and clopen. Then $X = f^{-1}(A) \cup f^{-1}(B)$ is a separation of X , which is a contradiction. Therefore $Y = f(X)$ is connected.

In the general case, apply the above to C with the relative topology and $f(C)$ with the relative topology. \square

As an application of the above, in a normed vector space X , convex sets are connected.

Recall that $C \subseteq X$ is convex if whenever $x, y \in C$, then the segment

$$\{(1-t)x + ty \mid 0 \leq t \leq 1\} \subseteq C.$$

The segment is connected (it is the continuous image of the interval $[0, 1]$). For $x_0 \in C$, $C = \bigcup_{x \in C} [x_0, x]$ which is connected by Proposition 1.10.