MATH 202A — LECTURE NOTES FOR AUG 31, 2005

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1. More topology

Let X be a topological space.

Definition 1.1 (Relative Topology)**.** If $Y \subseteq X$, $Y \neq \emptyset$, the relative topology on Y has as its open sets the sets $U \cap Y$, where U is an subset of X.

Definition 1.2 (Limit Point)**.** For $A \subseteq X$, we say $x \in X$ is a limit point (or accumulation point) of A if for every open U such that $x \in U$, $U \cap (A \setminus \{x\}) \neq \emptyset$. A' is the set of limit points of A.

Example 1.3. Let $X = \mathbb{R}$, $A = \mathbb{N}$, and $A' = \emptyset$. Another example is $A = \{\, 1/n \mid n \in \mathbb{N} \,\}$, where $A' = \{0\}$.

Definition 1.4 (Closure)**.** The closure $\overline{A}$ of A, is $A \cup A'$.

Proposition 1.5. *$\overline{A}$ is closed.*

Proof. Let $x \in X \setminus \overline{A}$. Then $x \notin A'$. So ther exists an open set U containing x such that $U \cap (A \setminus \{x\}) = \emptyset$. But $x \notin A$, so $U \subseteq X \setminus A$, since $x \notin A$. Since U is open, $U \cap A' = \emptyset$, so $U \cap \overline{A} = \emptyset$. So take the union of all such U. Thus $X \setminus \overline{A}$ is open. □

Proposition 1.6. *If F is closed, and $A \subseteq F$, then $\overline{A} \subseteq F$*

Proof. Since $X \setminus F$ is open and disjoint from A, it is disjoint from A', hence disjoint from $\overline{A}$. □

Proposition 1.7. *A is closed if and only if it contains all its limit points.*

Definition 1.8 (Density)**.** A is dense in X if $\overline{A} = X$.

Example 1.9. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 1.10. For $A \subseteq X$, we say x is a boundary point of A if for all U such that $x \in U$, $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) \neq \emptyset$. We call ∂A the set of all boundary points of A.

Properties

1. $\partial A = \partial(X \setminus A)$.
2. $\overline{A} = A \cup \partial A$.
3. A is closed if and only if $\partial A \subseteq A$.
4. A is open if and only if $\partial A \cap A = \emptyset$.

Definition 1.11 (Int)**.** int $A = A \setminus \partial A$. This is $A^{\bullet}$ in Folland

Properties

1. int $A = X \setminus \overline{X \setminus A}$.
2. int A is open.

Scribed by Chris Crutchfield

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2. Countability Axioms

2.1. **Axiom 0.** X is separable if it has a countable dense subset.

2.2. **Axiom 1.** X satisfies the first axiom of countability if each point of X has a countable neighborhood base.

Example 2.1. A metric space (X, ρ) is first countable. For $x \in X$, the balls $B_{1/n}(x)$ for $n \in \mathbb{N}$ form a neighborhood base at x.

2.3. **Axiom 2.** X satisfies the second axiom of countability if it has a countable base of open sets.

Example 2.2. If (X, ρ) is a separable metric space then it is second countable.

Proof.

$$\mathcal{B} = \left\{ B_{\frac{1}{n}}(a) \mid a \in A, n \in \mathbb{N} \right\}$$

is a countable set. Let $U \subseteq X$ be open, and $x \in U$. We want $B_{1/n}(a)$ such that $x \in B_{1/n}(a) \subseteq U$. Pick $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. Pick n such that $1/n < \varepsilon/2$. Pick $a \in A$ such that $\rho(x, a) < 1/n$. Then $x \in B_{1/n}(a)$. Let $y \in B_{1/n}(a)$. Then

$$\rho(y, x) \leq \rho(y, a) + \rho(a, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $B_{1/n}(a) \subseteq U$. □

Theorem 2.3 (Lindelöt's Theorem)**.** *Let X be second countable, and $\mathcal{U}$ a family of open subsets. Then there is a countable subfamily $\mathcal{U}_0 \subseteq \mathcal{U}$ with the same union.*

$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}_0} U$$

Proof. Let $\mathcal{B}$ be a countable base, and let $\mathcal{B}_0$ consist of all $V \in \mathcal{B}$ such that $V \subseteq U$ for some $U \in \mathcal{U}$. Then $\mathcal{B}_0$ is countable and $\bigcup_{V \in \mathcal{B}_0} V = \bigcup_{U \in \mathcal{U}} U$ (because every set is a union of basic open sets.

For $V \in \mathcal{B}_0$, pick $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Let $\mathcal{U}_0 = \{ U_V \mid V \in \mathcal{B}_0 \}$ is a countable set. In addition,

$$\begin{aligned} \bigcup_{U \in \mathcal{U}_0} U \supseteq \bigcup_{V \in \mathcal{B}_0} V &= \bigcup_{U \in \mathcal{U}} U \\ \bigcup_{U \in \mathcal{U}_0} U &= \bigcup_{U \in \mathcal{U}} U \end{aligned}$$

□