

## MATH 202A, ASSIGNMENT 9

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**Exercise (31).** Let  $f$  be a function in  $C[0, 1]$  such that

$$\int_0^1 x^n f(x) dx = 0$$

for all positive integers  $n > 2005$ . Prove  $f = 0$ .

*Proof.* Suppose that  $f$  is non-zero on  $[0, 1]$ . Then there must exist some interval  $[a, b] \subseteq [0, 1]$  such that  $f$  is non-zero on this interval (by continuity of  $f$ ). Let  $M = \inf \{ f(x) : x \in [a, b] \} > 0$ . Consider the continuous function  $g(x)$  defined by

$$g(x) = \begin{cases} 0 & x \notin [a, b] \\ \frac{2(x-a)}{b-a} & x \in [a, \frac{a+b}{2}] \\ \frac{2(b-x)}{b-a} & x \in [\frac{a+b}{2}, b] \end{cases}$$

By the Stone-Weierstrass theorem, there exists a sequence of polynomials  $(p_n)_1^\infty$  such that for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $|p_n(x) - g(x)| < \varepsilon$  for all  $x \in [0, 1]$ .

Note that  $g(x) \geq 1/2$  on the interval  $[(a+b)/4, 3(a+b)/4]$ . Thus:

$$\int_0^1 x^{2006} f(x) g(x) dx > \frac{a^{2006} M(a+b)}{4}$$

Let  $\varepsilon < (a^{2006} M(a+b))/4$ . Then for  $n > N$ ,

$$\int_0^1 x^{2006} f(x) p_n(x) dx > \frac{a^{2006} M(a+b)}{4} - \varepsilon > 0$$

But we know that  $\int_0^1 x^n f(x) dx = 0$ , so since  $p_n(x) = \sum_0^m a_k x^k$ :

$$\int_0^1 x^{2006} f(x) p_n(x) dx = \sum_{k=0}^m a_k \int_0^1 x^{2006+k} f(x) dx = 0$$

Which is a contradiction. □

**Exercise (32).** Prove there exists a sequence  $(p_n)_1^\infty$  of polynomials such that  $p_n(0) = 0$  for all  $n$ ,  $p_n \rightarrow 1$  uniformly on each compact subinterval of  $(0, \infty)$ , and  $p_n \rightarrow -1$  uniformly on each compact subinterval of  $(-\infty, 0)$ .

*Proof.* Consider the family of continuous functions

$$f_n(x) = \begin{cases} -1 & x \in (-\infty, -\frac{1}{n}) \\ nx & x \in [-\frac{1}{n}, \frac{1}{n}] \\ 1 & x \in (\frac{1}{n}, \infty) \end{cases}$$

For each  $f_n(x)$ , we consider the sequence of polynomials  $g_{n,m,k}$  converging uniformly to  $f_n$  on  $[-m, m]$  (by the Stone-Weierstrass Theorem), such that

$$|g_{n,m,k}(x) - f_n(x)| < \frac{1}{k}, \text{ for all } x \in [-m, m]$$

We aim to show that the sequence  $p_n(x) = g_{n,n,n}(x) - g_{n,n,n}(0)$  satisfies the above conditions.

Let  $[a, b]$  be a compact subinterval of  $\mathbb{R}$  such that  $0 < a$ . Let  $\varepsilon > 0$ . There exists an  $i \in \mathbb{N}$  such that  $1/i < \varepsilon$ . Then there also exists  $j \in \mathbb{N}$  such that  $1/j < a$ , and  $k \in \mathbb{N}$  such that  $k > b$ . Let  $N = \max\{2i, j, k\}$ . Then for any  $n > N$ ,

$$|p_n(x) - 1| = |g_{n,n,n}(x) - g_{n,n,n}(0) - 1| < 1/n, \text{ on the interval } [-n, n]$$

So  $(p_n(x))_1^\infty$  converges uniformly to 1 on any compact subinterval of  $(0, \infty)$ . A similar argument shows that  $(p_n(x))_1^\infty$  converges uniformly to -1 on any compact subinterval of  $(-\infty, 0)$ .

In addition  $p_n(0) = 0$  for all  $n$ , so this sequence of polynomials satisfies the given conditions.  $\square$

**Exercise (33).** Let  $X$  be a compact Hausdorff space.

- (a) Let  $D$  be a dense subset of  $C(X)$ . Prove the sets  $\{x \in X : |f(x)| < 1\}$  with  $f$  in  $D$  form a base for the topology of  $X$ . Deduce that  $X$  is second countable if  $C(X)$  is separable.

*Proof.* To show that these sets form a base for the topology of  $X$ , we show that for an arbitrary open set  $V \subseteq X$  there exists an  $f \in D$  such that  $U_f = \{x \in X : |f(x)| < 1\} \subseteq V$ . Clearly  $U_f$  is an open set by the continuity of  $f$ , so such a family  $\mathcal{U}$  of open sets would form a topological basis for  $X$ .

Let  $v \in V$ . Then we have that  $\{v\}$  is closed. By Urysohn's Lemma, there exists a continuous function  $g(x)$  such that  $g(v) = 0$  and  $g(V^c) = 2$ . Clearly, then, we know that  $U_g \subseteq V$ . We aim to find  $f \in D$  close enough to  $g$  so that  $U_f \subseteq V$  as well.

But this is clearly true since we can pick  $\varepsilon < 1$ , and find a  $f \in D$  such that  $\sup_{x \in X} |f(x) - g(x)| < \varepsilon$ . Hence  $f(V^c) > 1$  and  $|f(v)| < 1$ . Therefore  $U_f \subseteq V$  and  $\mathcal{U}$  forms a topological basis for  $X$ .

If  $C(X)$  is separable, then we let  $D$  be a countable dense subset of  $C(X)$ . We consider  $\mathcal{U} = \{U_f : f \in D\}$  where  $U_f = \{x \in X : |f(x)| < 1\}$ . By the above,  $\mathcal{U}$  forms a topological basis for  $X$ . Since  $\mathcal{U}$  is countable, it must be the case that  $X$  is second countable.  $\square$

- (b) Prove  $C(X)$  is separable if  $X$  is second countable.

*Proof.* By Urysohn's Metrization Theorem, we know that  $X$  is metrized by  $\rho$ . Consider the sets  $U_{x,n} = \{y \in X : \rho(x, y) < 1/n\}$ . If we fix  $n \in \mathbb{N}$ , then the set  $\mathcal{U}_n = \{U_{x,n} : x \in X\}$  forms an open cover of  $X$ . Since  $X$  is compact, there exists a finite subcover  $\mathcal{V}_n$ . We consider the countable family  $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ .

Consider the set of tuples  $(U, V)$  such that  $U, V \in \mathcal{V}$ ,  $U \subseteq V$ , and  $\overline{U} \cap V^c = \emptyset$ . For each such tuple, we can define by Urysohn's Lemma a continuous function  $f_{U,V}$  such that  $f_{U,V}(\overline{U}) = 1$  and  $f_{U,V}(V^c) = 0$ . We consider the algebra generated by these  $f_{U,V}$  over  $\mathbb{Q}$ . That is to say, we consider finite rational linear combinations of the product of finitely many functions  $f_{U,V}$ . Let's call this algebra  $A$ , which is a countable subalgebra of  $C(X)$ . Our goal is to use

the Stone-Weierstrass Theorem to prove that  $\overline{A} = C(X)$ . We first must show that  $\overline{A}$  separates points of  $X$ .

Clearly  $A$  separates points of  $X$ . Let  $x, y \in X$ . Then since  $X$  is Hausdorff, there exists disjoint neighborhoods  $U, V$  of  $x$  and  $y$  respectively. But there exists  $U_{x,n} \in \mathcal{V}$  such that  $U_{x,n} \subsetneq U$ . Therefore we consider  $f_{U_{x,n}, U}(x)$ . By construction, this function separates  $x$  and  $y$ . Therefore  $\overline{A}$  must separate points of  $X$  as well.

In addition, we must show that there exists a nonvanishing function in  $\overline{A}$ . Let  $n$  be fixed. We consider the set of functions

$$\{f_{U, \{y\}^c} : U \in \mathcal{V}_n, y \notin \overline{U}\}.$$

Taking the sum of these functions results in a nonvanishing function on  $X$ .

Therefore, by the Stone-Weierstrass Theorem,  $\overline{A} = C(X)$ . Since  $A$  is a countable set, this is what we wanted to prove.  $\square$