MATH 202A, ASSIGNMENT 7

CHRIS CRUTCHFIELD

Exercise 1. *Let $X = \prod_{i\in\mathcal{I}} X_i$ be a product space, and for each i let A_i be a nonempty subset of X_i. Let $A = \prod_{i\in\mathcal{I}} A_i$. Prove that*

$$\overline{A} = \prod_{i\in\mathcal{I}} \overline{A_i}.$$

Proof. Let x be a limit point of A. Then we want to show that $\pi_i(x)$ is contained in $\overline{A_i}$ for all $i \in \mathcal{I}$.

Fix $i \in \mathcal{I}$. Let N_i be an arbitrary neighborhood of $\pi_i(x)$. Then $M_i = \pi_i^{-1}(N_i)$ is a neighborhood of x. Since x is a limit point of A, there exists $y_i \neq x$, $y_i \in A \cap M_i$. Therefore $\pi_i(y_i) \in A_i \cap N_i$. Since N_i was chosen arbitrarily, we see that $\pi_i(\overline{A}) \subseteq \overline{A_i}$. Since i was chosen arbitrarily, we have that $\overline{A} \subseteq \prod_{i\in\mathcal{I}} \overline{A_i}$.

Now we prove the converse. Suppose that x_i is a limit point of A_i for all $i \in \mathcal{I}$. Let $x = \prod_{i\in\mathcal{I}} x_i$. Then for all N a neighborhood of x, we have that $\pi_i(N)$ is a neighborhood of x_i for all $i \in \mathcal{I}$. Thus since x_i is a limit point of A_i for all i, then there exists $y_i \neq x_i$ such that $y_i \in A_i \cap \pi_i(N)$. We let $y = \prod i \in \mathcal{I} y_i$. Thus $y \in N \cap A$, whence $\prod_{i\in\mathcal{I}} \overline{A_i} \subseteq \overline{A}$. □

Exercise 2. *Prove that the topological space X is Hausdorff if and only if the diagonal in $X \times X$, the set $D = \{\,(x, x) : x \in X\,\}$, is closed in the product topology.*

Proof. First we suppose that X is Hausdorff and we aim to show that the diagonal D in $X \times X$ is closed. Suppose (x, y) is a limit point of D and $x \neq y$. Then since X is Hausdorff, there exists N_1 and N_2 disjoint such that $x \in N_1$, $y \in N_2$. Now we consider $N_1 \times N_2$. Since this is a neighborhood of (x, y), there exists $(z, z) \in N_1 \times N_2$. But this implies that $z \in N_1 \cap N_2$, which is a contradiction. Therefore D is closed.

Now suppose that the diagonal in $X \times X$ is closed. Let $(x, y) \in X \times X$, such that $x \neq y$. Since the diagonal D is closed, there exists a neighborhood N of (x, y) such that N is disjoint from D. Therefore we have that $(x, x) \notin N$ and $(y, y) \notin N$. Applying π_i, we get that $x \notin \pi_2(N)$ and $y \notin \pi_1(N)$. Now we suppose that $z \in \pi_1(N) \cap \pi_2(N)$. Then this implies that $(z, z) \in N$, which is a contradiction. Therefore we have that $x \in \pi_1(N)$, $y \in \pi_2(N)$, and $\pi_1(N) \cap \pi_2(N) = \emptyset$. Hence X is Hausdorff. □

Exercise 3. *Prove the following:*

(a) The space $\mathbb{R}^\mathbb{R}$ is separable.
(b) The space $\mathbb{R}^\mathbb{R}$ is not first countable.

(a) *Proof.* We let $A = \{A_1, A_2, \dots\}$ be a countable base for $\mathbb{R}$, since $\mathbb{R}$ is second countable. Then we consider $\mathcal{I}$ a finite subset of $\mathbb{N}$ such that $\{A_i\}_{i\in\mathcal{I}}$ are

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pairwise disjoint. Let $q \in \mathbb{Q}^{\mathcal{I}}$. Then we define $b_{\mathcal{I},q} = \prod_{x\in\mathbb{R}} c_x$, where c_x is $\pi_i(q)$ if $x \in A_i$ and 0 if $x \notin A_i$ for all $i \in \mathcal{I}$. Now, since $\mathbb{Q}^{\mathcal{I}}$ is countable, the set

$$B_{\mathcal{I}} = \bigcup_{q\in\mathbb{Q}^{\mathcal{I}}} b_{\mathcal{I},q}$$

is countable as well. Since the set of finite subsets of $\mathbb{N}$ is countable, then if $\mathcal{J} = \{\, \mathcal{I} \subseteq \mathbb{N} : \mathcal{I} \text{ finite}, \{A_i\}_{i\in\mathcal{I}} \text{ pairwise disjoint}\,\}$.

$$B = \bigcup_{\mathcal{I}\in\mathcal{J}} B_{\mathcal{I}}$$

is countable as well. Now we show that this set is dense in $\mathbb{R}^{\mathbb{R}}$.

Let $x \in \mathbb{R}^{\mathbb{R}}$ and N be a neighborhood of x. Then by the definition of the product topology, we must have that there exists some finite subset $\mathcal{I} \subseteq \mathbb{R}$, such that $\pi_i(N)$ is a proper open subset of $\mathbb{R}$ for all $i \in \mathcal{I}$, and $\pi_i(N) = \mathbb{R}$ for all $i \notin \mathcal{I}$. Since $\mathbb{R}$ is Hausdorff, and since A is a countable base, there exists some map $f : \mathcal{I} \to \mathbb{N}$ such that for all $i \in \mathcal{I}$, $i \in A_{f(i)}$ and $\{A_{f(i)}\}_{i\in\mathcal{I}}$ is pairwise disjoint. Also, for each $i \in \mathcal{I}$ there exists $q_{f(i)} \in \mathbb{Q}$ such that $q_{f(i)} \subseteq \pi_i(N)$, by the density of $\mathbb{Q}$ in $\mathbb{R}$. Letting $q = \prod_{j\in f(\mathcal{I})} q_j$, we see that $b_{f(\mathcal{I}),q}$ is by definition contained in N. But $b_{f(\mathcal{I}),q} \in B$. Therefore B is a countable dense subset of $\mathbb{R}^{\mathbb{R}}$. □

(b) *Proof.* Suppose that $\mathbb{R}^{\mathbb{R}}$ is first countable. Then we fix $x \in \mathbb{R}^{\mathbb{R}}$. By assumption there exists a countable set $\{U_1, U_2, U_3, \dots\}$ of neighborhoods of x such that for all neighborhoods V of x, there exists $n \in \mathbb{N}$ such that $U_n \subseteq V$. However, since each U_i is a neighborhood of x, there exists some $\mathcal{I}_i$ a finite subset of $\mathbb{N}$ such that for all $j \in \mathcal{I}_i$, $\pi_j(U_i)$ is a proper open subset of $\mathbb{R}$, and for all $j \notin \mathcal{I}_i$, $\pi_j(U_i) = \mathbb{R}$. But we notice that $A = \bigcup_{i\in\mathbb{N}} \mathcal{I}_i$ is a countable subset of $\mathbb{R}$. Since $\mathbb{R}$ is uncountable, there exists $b \notin A$. Letting $V = \pi_b^{-1}(Y)$, where Y is the open interval $(0,1)$, we see that there cannot exist an $n \in \mathbb{N}$ such that $U_n \subseteq V$, because that would imply that $\pi_b(U_n) = \mathbb{R} \subseteq (0,1)$. □

Exercise 4. *Let X be the product space $\{0,1\}^{\mathbb{N}}$, where $\{0,1\}$ has the discrete topology. Thus, X is a compact Hausdorff space. Define the function $\varphi : X \to \mathbb{R}$ by*

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{2x_n}{3^n},$$

and note that $\varphi(X) = C$, the Cantor set.

(a) Prove φ is continuous.
(b) Prove φ is one-to-one.
(c) Deduce that φ is a homeomorphism of X onto C.
(d) Prove C is homeomorphic to $C \times C$.
(e) Prove C is homeomorphic to $C^{\mathbb{N}}$.

(a) *Proof.* For this problem we will use the definition of continuity where if for all V a neighborhood of $f(x)$ there exists U a neighborhood of x such that $f(U) \subseteq V$, then f is continuous at x.

Let $x \in \{0,1\}^{\mathbb{N}}$. Let V be a neighborhood of $\varphi(x)$. Now, we clearly have that the open sets $\{(0,1/3), (2/3,1), (0,1/9), (2/9,1/3), \dots\}$ used in the construction of the Cantor set form a topological basis for C. Hence there exists some interval U of this form contained in V. But we note that in each such interval, there exists $n \in \mathbb{N}_+$ such that for all x in the interval and for all $i \leq n$, $\pi_i(x)$ is a singleton set (namely $\pi_i(\varphi^{-1}(\inf U))$), and for all $i > n$, $\pi_i(x) = \{0, 1\}$. Therefore, $\varphi^{-1}(U)$ is an open set in X. Therefore φ is continuous at x. Since our choice of x was arbitrary, φ is a continuous function. □

(b) *Proof.* Suppose that $x, y \in \{0, 1\}^{\mathbb{N}}$, $x \neq y$, such that $\varphi(x) = \varphi(y)$. Comparing digits of x and y, we see that since $x \neq y$, there exists some smallest $n \in \mathbb{N}$ such that $x_n \neq y_n$, where $x_n = \pi_n(x)$ and so on. Without loss of generality, let's suppose $x_n = 1$ and $y_n = 0$. In the worst case, $x_k = 0$ for all $k > n$ and $y_k = 1$ for all $k > n$. In this case, $\varphi(x) - \varphi(y) = 2/3^n - \sum_{k=n+1}^{\infty} 2/3^k = 2/3^n - 1/3^n = 1/3^n \neq 0$. Therefore φ is one-to-one. □

(c) *Proof.* By the Tychonoff Product Theorem, X is a compact set (since each $\{0, 1\}$ is trivially compact). Since by (a), φ is continuous, and since by (b), φ is a bijection, and since the Cantor set is a Hausdorff space (by virtue of being contained in a Hausdorff space), by one of the properties of compactness presented in class, φ is a homeomorphism. □

(d) *Proof.* Consider the map $\sigma : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$, which takes the element $(x_1, x_2, x_3, \ldots)$ to $((x_1, x_3, x_5, \ldots), (x_2, x_4, x_6, \ldots))$. This map is clearly a bijection. It is continuous because if U is an open set in the image space, then there are some subsets of $\mathbb{N}$, $\mathcal{I}_1$ and $\mathcal{I}_2$ such that for all $i \notin \mathcal{I}_1 \cup \mathcal{I}_2$, $\pi_1(\pi_i(U))$ or $\pi_2(\pi_i(U))$ is $\{0, 1\}$. Therefore the preimage of U is an open set, with $\mathcal{I}_1 \cup \mathcal{I}_2$ as its set of indices of proper open subsets. In the same it is clearly true that the inverse function is continuous.

We construct a homeomorphism from C to $C \times C$ as follows. Let the homeomorphism be $(\varphi, \varphi) \circ \sigma \circ \varphi^{-1}$. □

(e) *Proof.* There exists a bijection ρ between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. We aim to construct a homeomorphism σ from $\{0, 1\}^{\mathbb{N}}$ to $(\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$. Let $x = (x_1, x_2, x_3, \ldots)$ and let $\sigma(x) = \prod_{i \in \mathbb{N}} \prod_{j \in \mathbb{N}} x_{\rho^{-1}(i,j)}$. Clearly σ is a bijection, since ρ is a bijection. Let N be an open set in $\{0, 1\}^{\mathbb{N}}$. Then there is some finite subset $\mathcal{I} \subseteq \mathbb{N}$ where $\pi_i(N)$ is a proper subset of $\{0, 1\}$ on $\mathcal{I}$, and the entire set elsewhere. Clearly, then, in the image, there is some finite subset $\mathcal{J} \subseteq \mathbb{N}$ such that $\pi_j(\sigma(N))$ is a proper subset of $\{0, 1\}^{\mathbb{N}}$, and the entire space elsewhere. The same argument holds for the opposite direction, ensuring that σ is indeed a homeomorphism.

Now we construct a homeomorphism from C to $C^{\mathbb{N}}$ as follows. Let the homeomorphism be $\left(\prod_{i \in \mathbb{N}} \varphi\right) \circ \sigma \circ \varphi^{-1}$. □