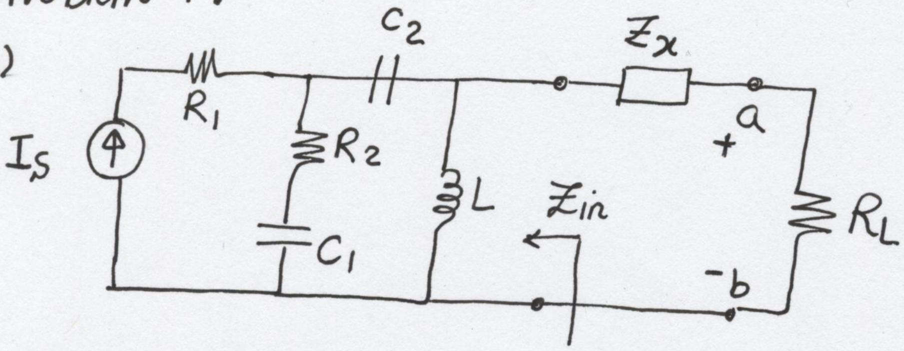
# Midterm 2 Solutions:

Problem 1:

a)




$$Z_1 = Z_{C_1} + R_2 + Z_{C_2}$$

$$Z_1 = \frac{1}{j \times 10^6 \times 10^{-6}} + 1 + \frac{1}{j \times 10^6 \times 10^{-6}} = 1 - 2j$$

$$Z_L = j\omega L = j \times 10^6 \times 10^{-6} = j$$

$$Z_{in} = Z_1 \parallel Z_L = \frac{j(1-2j)}{j + 1 - 2j} = \frac{2+j}{1-j} = \frac{(2+j)(1+j)}{2}$$

$$= \frac{1}{2} + \frac{3}{2}j$$

$$Z_x = -Im\{Z_{in}\} = -\frac{3}{2}j$$

b) $Z_{th} = \frac{1+3j}{2} + Z_x$

$$Z_x = -\frac{3}{2}j$$

Imag. part $Z_{th}$ should be zero.

Real part of $Z_{th}$ should be minimized.

Problem 2:

a)

$$Z_{i_1} = R_{i_1} \parallel Z_{C_{i_1}} = \frac{R_{i_1} \times \frac{1}{j\omega C_{i_1}}}{R_{i_1} + \frac{1}{j\omega C_{i_1}}} = \frac{R_{i_1}}{1 + j\omega R_{i_1} C_{i_1}}$$

$$\frac{V_1(\omega)}{V_S(\omega)} = \frac{Z_{i_1}}{R_S + Z_{i_1}} = \frac{R_{i_1}}{(R_{i_1} + R_S) + j\omega R_S R_{i_1} C_{i_1}}$$

b) $$Y_{eq} = \frac{1}{R_{o_1}} + \frac{1}{R_{i_2}} + j\omega (C_{o_1} + C_{i_2}) = \frac{(R_{i_2} + R_{o_1}) + j\omega R_{i_2} R_{o_1} (C_{o_1} + C_{i_2})}{R_{o_1} R_{i_2}}$$

$Z_{eq} = \frac{1}{Y_{eq}}$

$V_2 = -g_{m_1} V_1 Z_{eq}$

$$\frac{V_2}{V_1} = \frac{-g_{m_1} R_{o_1} R_{i_2}}{(R_{i_2} + R_{o_1}) + j\omega R_{i_2} R_{o_1} (C_{o_1} + C_{i_2})}$$

c) $$Z_{o_2} = R_{o_2} \parallel \frac{1}{j\omega C_{o_2}} = \frac{R_{o_2}}{1 + j\omega R_{o_2} C_{o_2}}$$

$V_{out} = g_{m_2} V_2 Z_{o_2}$

$$\frac{V_{out}(\omega)}{V_2(\omega)} = \frac{g_{m_2} R_{o_2}}{1 + j\omega R_{o_2} C_{o_2}}$$

d) $$\frac{V_{out}(\omega)}{V_S(\omega)} = \frac{V_1(\omega)}{V_S(\omega)} \cdot \frac{V_2(\omega)}{V_1(\omega)} \cdot \frac{V_{out}(\omega)}{V_2(\omega)}$$

$$\frac{V_{out}(\omega)}{V_S(\omega)} = \frac{R_{i_1}}{(R_{i_1} + R_S) + j\omega R_S R_{i_1} C_{i_1}} \cdot \frac{-g_{m_1} R_{o_1} R_{i_2}}{R_{i_2} + R_{o_1} + j\omega R_{i_2} R_{o_1} (C_{o_1} + C_{i_2})} \cdot \frac{g_{m_2} R_{o_2}}{1 + j\omega R_{o_2} C_{o_2}}$$

Problem 3:

a)

$$\frac{V_{S1}}{R_1} = \frac{-V_{o1}}{R_2} - \frac{V_{out}}{R_3}$$

$$\frac{V_{o1}}{R_5} = \frac{-V_{out}}{R_4} \Rightarrow V_{o1} = \frac{-R_5}{R_4} V_{out}$$

$$\frac{V_{S1}}{R_1} = \frac{R_5}{R_2 R_4} V_{out} - \frac{V_{out}}{R_3}$$

$$\frac{V_{out}}{V_{S1}} = \frac{R_2 R_3 R_4}{R_1 (R_3 R_5 - R_2 R_4)}$$

b) $V_{out}(0) = \left(\frac{-R_2}{R_1}\right)\left(\frac{-R_4}{R_5}\right) = \frac{R_2 R_4}{R_1 R_5}$

$V_{out}(\infty) = 0$

$$\frac{V_{S1}}{R_1} + \frac{V_{o1}}{R_2} + \frac{C dV_{out}}{dt} = 0$$

$V_{o1} = \frac{-R_5}{R_4} V_{out}$ $\qquad$ $V_{S1}(t>0) = 0$

$$\Rightarrow \frac{-R_5}{R_4 R_2} V_{out} + \frac{C dV_{out}}{dt} = 0$$

$$\frac{dV_{out}}{dt} - \frac{R_5}{C R_4 R_2} V_{out} = 0 \Rightarrow \tau = \frac{-R_4 R_2 C}{R_5}$$

$$\Rightarrow V_{out}(t) = \frac{R_2 R_4}{R_1 R_5} e^{\frac{R_5 t}{R_2 R_4 C}}$$

c)

$$\frac{V_{out}(\omega)}{V_{S_1}(\omega)} = \frac{R_2 Z_3 R_4}{R_1(Z_3 R_5 - R_2 R_4)} = \frac{1}{1 - 318\pi j}$$

$V_{S_1}(\omega) = 1$

$$\Rightarrow V_{out}(\omega) = \frac{1}{1-318\pi j} = \frac{1}{\sqrt{1+(318\pi)^2}} \angle \tan^{-1}(318\pi)$$

$$v_{out}(t) = \text{Real}\left\{V_{out}\, e^{j\omega t}\right\} = \frac{1}{\sqrt{1+(318\pi)^2}} \cos(318 \times 10^2 \pi t + \tan^{-1}(318\pi))$$

Problem 4:

$V_C(0) = 0 \Rightarrow V_R(0) = 1^V$

$V_R(\infty) = 0$ $\tau = RC$

$$V_R(t) = V_R(\infty) + [V_R(0) - V_R(\infty)]e^{-\frac{t}{\tau}}$$

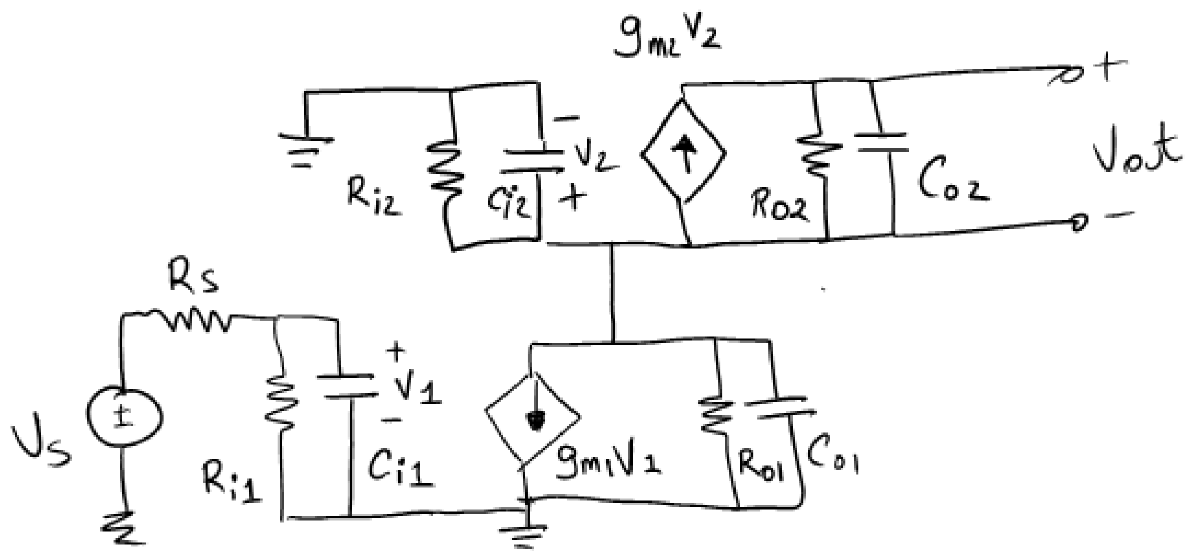
$$V_R(t) = \begin{cases} e^{-\frac{t}{\tau}} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

“They were watching, out there past men's knowing, where stars are drowning and whales ferry their vast souls through the black and seamless sea.”

**— Blood Meridian, or the Evening Redness in the West (***Cormac McCarthy***)**

**Problem 2** *Transfer functions* (25 points)

Consider the circuit below.



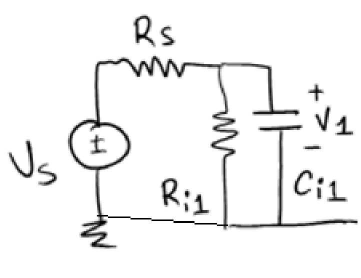

Assume $v_S(t) = v_S cos(\omega t)$

a) Find an expression for $\frac{V_1(\omega)}{V_S(\omega)}$ *(5 points)*

**Solution:**

Since our input signal is a sinusoidal function, we can transform the circuit from time domain to phasor domain.

In order to find the transfer function (of the first stage) from the input, Vs(w), to the output, V1(w), we only consider the following part of circuit:




By inspection, this is a simple first-order low-pass filter.

Since any transfer function can be decomposed into the frequency-dependent part and the frequency-independent part (constant).

Recognizing that for a simple first-order low-pass filter, the frequency-independent part, or the constant is the DC gain of the circuit, and the frequency-dependent part is given by the canonical form,

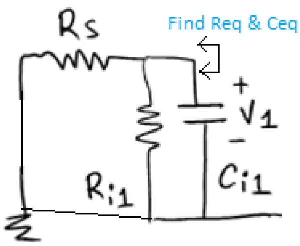
$$\frac{1}{1 + j\omega\tau}$$

We need to find the DC gain and the time constant ($\tau$).

In order to determine the DC gain, we open-circuit the capacitor (because capacitors act like an open circuit at DC). And we find that the relationship between Vs(w = 0) and $V_1$(w = 0) is given by a voltage divider between Rs and $R_{i1}$.

$$\left.\frac{V_1(\omega)}{V_s(\omega)}\right|_{\omega=0} = \frac{R_{i1}}{R_{i1} + R_s}$$

In order to find the time constant, we turn off all the independent sources and find the parallel resistance and capacitance across node $V_1$ and GND.

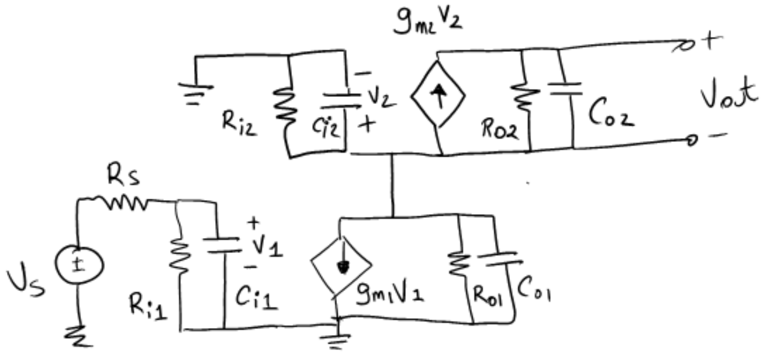



Here, $R_{eq1} = R_s || R_{i1} = \frac{R_s R_{i1}}{R_s + R_{i1}}$ and $C_{eq1} = C_{i1}$

Therefore, $\tau_1 = R_{eq1} C_{eq1} = \frac{R_s R_{i1}}{R_s + R_{i1}} C_{i1}$

Putting everything together,

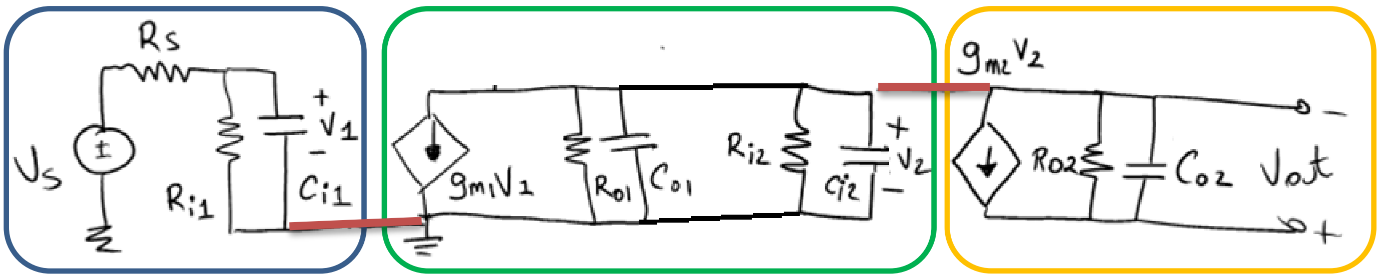
$$\frac{V_1(\omega)}{V_s(\omega)} = \left.\frac{V_1(\omega)}{V_s(\omega)}\right|_{\omega=0} \cdot \frac{1}{1 + j\omega\tau_1} = \frac{R_{i1}}{R_{i1} + R_s} \cdot \frac{1}{1 + j\omega \frac{R_s R_{i1}}{R_s + R_{i1}} C_{i1}}$$




b) Find an expression for $\dfrac{V_2(\omega)}{V_1(\omega)}$ *(10 points)*

**Solution:**

Re-drawing the circuit as the following makes the analysis clearer.




With this circuit, it is clear that you can separate the circuit into three stages. ($1^{st}$ – $2^{nd}$ – $3^{rd}$)

That is, there is no current through the wires that connect different stages (marked in red).

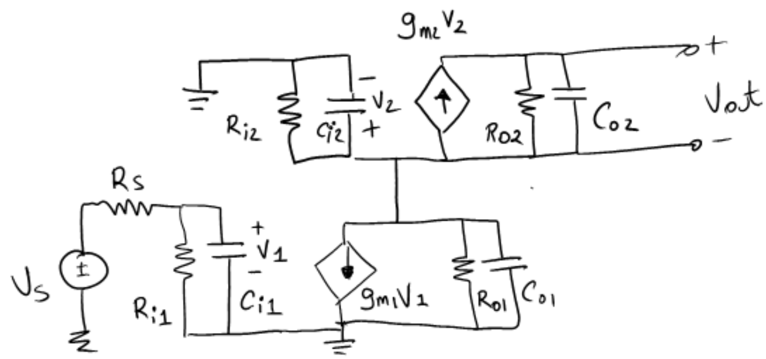
Following the technique from part a, we examine the DC gain of the second stage to be:

$$\left.\frac{V_2(\omega)}{V_1(\omega)}\right|_{\omega=0} = -g_{m1}(R_{o1}||R_{i2}) = \frac{-g_{m1}R_{o1}R_{i2}}{R_{o1}+R_{i2}}$$

Similarly, $\tau_2 = R_{eq2}C_{eq2} = (R_{o1}||R_{i2})(C_{o1}||C_{i2}) = \dfrac{R_{o1}R_{i2}}{R_{o1}+R_{i2}}(C_{o1}+C_{i2})$

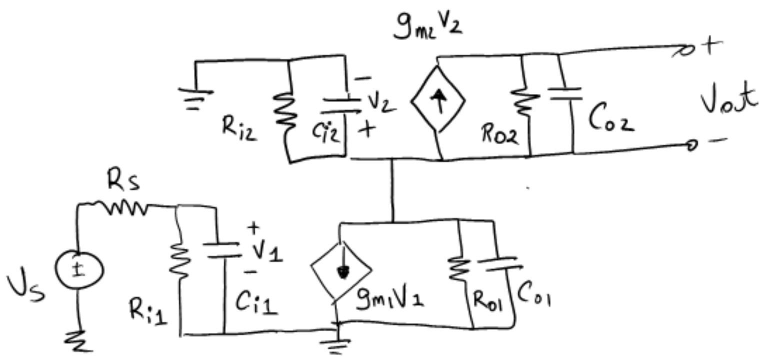
Therefore,

$$\frac{V_2(\omega)}{V_1(\omega)} = \left.\frac{V_2(\omega)}{V_1(\omega)}\right|_{\omega=0} \cdot \frac{1}{1+j\omega\tau_2} = \frac{-g_{m1}R_{o1}R_{i2}}{R_{o1}+R_{i2}} \cdot \frac{1}{1+j\omega\dfrac{R_{o1}R_{i2}}{R_{o1}+R_{i2}}(C_{o1}+C_{i2})}$$




c) Find an expression for $\frac{V_{out}(\omega)}{V_2(\omega)}$ *(5 points)*

| **Solution:** |
|---|
| The third stage gain is similar to the previous part, except simpler. $$\left.\frac{V_{out}(\omega)}{V_2(\omega)}\right\|_{\omega=0} = g_{m2}R_{o2}$$ Similarly, $\tau_3 = R_{eq3}C_{eq3} = R_{o2}C_{o2}$ <br><br> Therefore, $$\frac{V_2(\omega)}{V_1(\omega)} = \left.\frac{V_2(\omega)}{V_1(\omega)}\right\|_{\omega=0} \cdot \frac{1}{1+j\omega\tau_3} = g_{m2}R_{o2} \cdot \frac{1}{1+j\omega R_{o2}C_{o2}}$$ |




d) Find an expression for $\boldsymbol{\frac{V_{out}(\omega)}{V_S(\omega)}}$ *(5 points)*

**Solution:**

The total transfer function from Vs(w) to Vout(w) can be found by cascading the transfer function of each stage:

$$\frac{V_{out}(\omega)}{V_s(\omega)} = \frac{V_1(\omega)}{V_s(\omega)} \cdot \frac{V_2(\omega)}{V_1(\omega)} \cdot \frac{V_{out}(\omega)}{V_2(\omega)}$$

$$\frac{V_{out}(\omega)}{V_s(\omega)} = \frac{R_{i1}}{R_{i1}+R_s} \cdot \frac{1}{1+j\omega\frac{R_sR_{i1}}{R_s+R_{i1}}C_{i1}} \cdot \frac{-g_{m1}R_{o1}R_{i2}}{R_{o1}+R_{i2}} \cdot \frac{1}{1+j\omega\frac{R_{o1}R_{i2}}{R_{o1}+R_{i2}}(C_{o1}+C_{i2})} \cdot g_{m2}R_{o2} \cdot \frac{1}{1+j\omega R_{o2}C_{o2}}$$

Solution by Yun Jae Cho