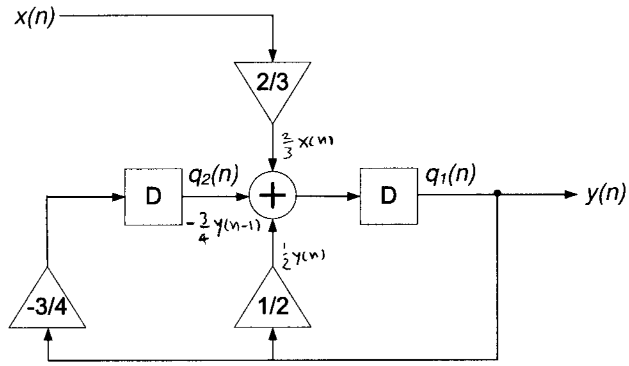
EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 2
Department of Electrical Engineering and Computer Sciences 23 March 2006
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name ______________ FIRST Name Solution

Lab Time ______________

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This examshould take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT2.1 (15 Points) A real, causal, discrete-time linear time-invariant (LTI) system is characterized by the following delay-adder-gain block diagram. The input and output at time n are denoted by $x(n)$ and $y(n)$, respectively. Each block D corresponds to a delay by one sample; that is, if the input to the delay block D is a signal r, the output of the delay block is the signal v, where $v(n) = r(n-1)$ for all n.



5 (a) Determine the linear, constant-coefficient difference equation that governs the input-output behavior of the system.

$$y(n) = \frac{1}{2}y(n-1) - \frac{3}{4}y(n-2) + \frac{2}{3}x(n-1)$$

10 (b) The outputs of the delay blocks are selected as the state variables $q_1(n)$ and $q_2(n)$, as shown in the figure. For this selection of state variables, determine the corresponding $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$ state-space representation of the LTI system.

$y(n) = q_1(n)$

Consider each delay block

$q_1(n+1) = q_2(n) + \frac{1}{2}q_1(n) + \frac{2}{3}x(n)$

$q_2(n+1) = -\frac{3}{4}q_1(n)$

Put these in matrix form

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{3}{4} & 0 \end{bmatrix}}_{A} \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \underbrace{\begin{bmatrix} 2/3 \\ 0 \end{bmatrix}}_{B} x(n)$$

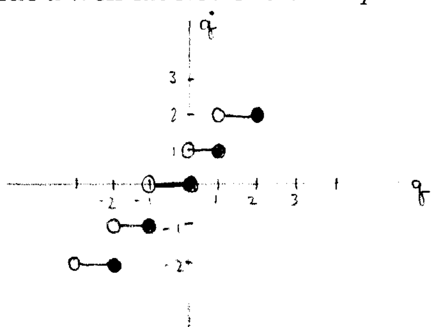
$$y(n) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \underbrace{0}_{D} \; x(n)$$

MT2.2 (30 Points) Consider a causal, autonomous system described in part by the following state-evolution equation:

$$\dot{q} = f(q) .$$

The state $q \in \mathbb{R}$. The function $f(q) = \lceil q \rceil$, where $\lceil q \rceil$ is the ceiling function, defined as the smallest integer not less than q. For example, $\lceil 1.2 \rceil = 2$ and $\lceil -2.3 \rceil = -2$.

10 (a) Provide a well-labeled sketch of $\dot{q}$ as a function of q.

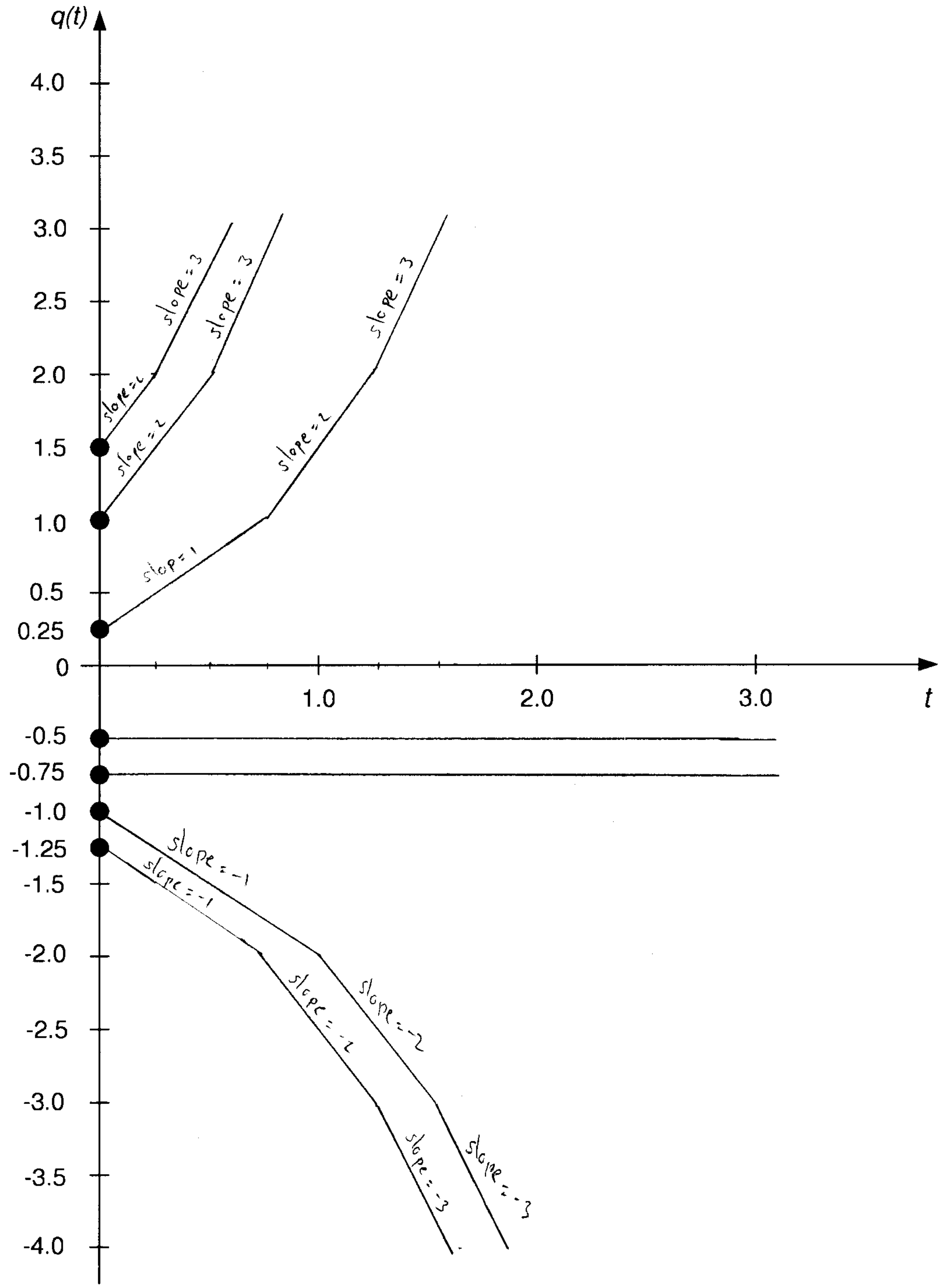


4 (b) Determine all the fixed points.

Solve $f(q^*) = \lceil q^* \rceil = 0$

$q^* \in (-1, 0]$

16 (c) Provide a well-labeled phase portrait of the system on the diagram of the next page. The phase portrait you provide must include the qualitatively-distinct trajectories based at the corresponding initial states q_0 specified in the diagram by filled-in dark small circles.


q(t)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.25
0
-0.5
-0.75
-1.0
-1.25
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
1.0
2.0
3.0
t
slope = 3
slope = 3
slope = 3
slope = 2
slope = 2
slope = 2
slope = 1
slope = -1
slope = -1
slope = -2
slope = -2
slope = -3
slope = -3

MT2.3 (30 Points) Consider a real, causal, second-order autonomous system described by the state-evolution equation

$$\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{q},$$

where

$$\boldsymbol{A} = \begin{bmatrix} -1 & \mu \\ +1 & -1 \end{bmatrix}.$$

The tunable parameter $\mu \in \mathbb{R}$. Let $(\lambda_1, \boldsymbol{v}_1)$ and $(\lambda_2, \boldsymbol{v}_2)$ denote the modes of the system. Note that depending on μ, the system may not have a second eigenvector $\boldsymbol{v}_2$, but we will not concern ourselves with that possibility.

6 (a) If $\mu = 1$, the eigenvalues of the system are $\lambda_1 = 0$ and $\lambda_2 = -2$. Determine the corresponding eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

$\lambda_1 = 0$: $(A - \lambda_1 I) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -2$: $(A - \lambda_2 I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_2 = 0 \Rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

or any scaled version of these

7 (b) If $\mu = 1$ and the initial state of the system is

$$\boldsymbol{q}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

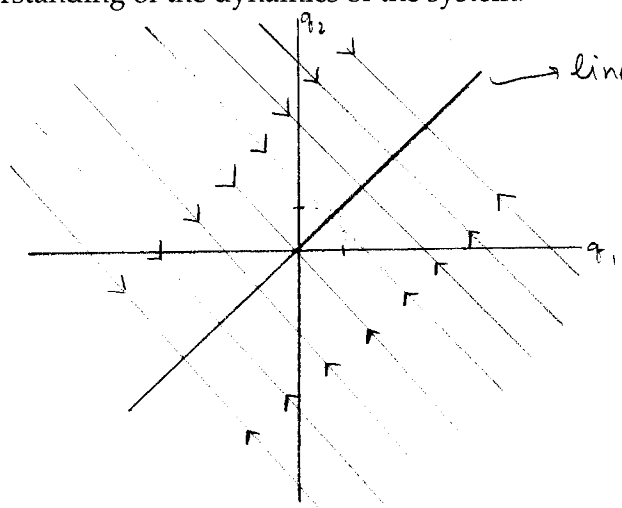
determine an expression for $\boldsymbol{q}(t)$ (the instantaneous state of the system) in terms of λ_1, λ_2, $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$. There should not be any free variables or coefficients in your expression.

$q(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

$q(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow c_1 = 1, \; c_2 = 1$

$q(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

8 (c) If $\mu = 1$, determine all the fixed points, and provide a well-labeled phase portrait, of the system. You need *not* classify the stability of the fixed points; instead, ensure that your phase portrait is an accurate indicator of your understanding of the dynamics of the system.



9 (d) Determine every value of μ for which the system's state trajectory is

(i) purely oscillatory.

(ii) unstable (growing) spiral.

(iii) stable (decaying) spiral.

Consider each of the cases (i), (ii), and (iii) separately. If no value of μ exists consistent with a particular case, state so. Regardless, explain your reasoning for each case succinctly, but clearly and convincingly.

Solve for eigenvalues in term of μ : $|A-\lambda I| = \begin{vmatrix} -1-\lambda & \mu \\ 1 & -1-\mu \end{vmatrix} = 0 \Rightarrow \lambda = -1 \pm \sqrt{\mu}$

(i) Not possible. Need purely imaginary λ, but here $Re\{\lambda\} = -1$.

(ii) Not possible. Need complex conjugate λ with positive real part
Again, here $Re\{\lambda\} = -1$

(iii) Need complex conjugate λ with negative real part.
This happens when $\mu < 0$

MT2.4 (18 Points) Consider a causal, discrete-time, single-input single-output (SISO) system whose $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$ state-space representation includes the state-update equation

$$\underbrace{\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix}}_{\boldsymbol{q}(n+1)} = \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix}}_{\boldsymbol{q}(n)} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{B}} x(n)$$

and the output equation

$$y(n) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\boldsymbol{C}} \boldsymbol{q}(n) .$$

Note that $\boldsymbol{D} = 0$ for this system.

The input signal, the state response, and the output response are $x : \mathbb{N}_0 \to \mathbb{R}$, $\boldsymbol{q} : \mathbb{N}_0 \to \mathbb{R}^2$, and $y : \mathbb{N}_0 \to \mathbb{R}$, respectively.

Throughout this problem, assume the initial state is zero (i.e., $\boldsymbol{q}(0) = \boldsymbol{0}$).

6 (a) Determine the modes $(\lambda_1, \boldsymbol{v}_1)$ and $(\lambda_2, \boldsymbol{v}_2)$ of the system, and explain why the system is unstable.

Since A is diagonal, we can readily get the modes (eigenvalue, eigenvector)

$\lambda_1 = \frac{1}{2}$ $\quad V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 3$ $\quad V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The system is unstable because the 2nd mode is unstable; $|\lambda_2| = 3 > 1$

6 (b) Determine the impulse response of the system, that is, the output response sample values $y(n)$ if the input x is the discrete-time unit impulse (Kronecker delta) function: $x(n) = \delta(n)$. Your final answer should be in terms of the mode parameters λ_1, λ_2, v_1, and v_2.

In lab, we have solved for zero-initial-state impulse response to

$$h(n) = \begin{cases} D & , n = 0 \\ CA^{n-1}B & , n > 0 \end{cases}$$

So for $n > 0$, $h(n) = CA^{n-1}B = [1 \quad 0] \begin{bmatrix} (1/2)^{n-1} & 0 \\ 0 & 3^{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{1}{2}\right)^{n-1} = \lambda_1^{n-1}$

$$h(n) = \begin{cases} 0 & , n = 0 \\ \lambda_1^{n-1} & , n > 0 \end{cases}$$

6 (c) Your friend Fran puts an opaque box around the system, and allows you only to apply an arbitrary input signal x to the system and measure the corresponding output response y. Fran does not allow you unfettered access to measure the state variables (i.e., you cannot peek inside the box; you can only perform input-output measurements and analysis of the system).

Fran claims that at some point in time you will (quite unexpectedly) see smoke billowing from the box, because you can never detect the presence of the unstable mode in the output response; that is, you cannot detect that a state variable is growing as $n \to \infty$. Is Fran correct? Explain your reasoning succinctly, but clearly and convincingly.

Fran is correct. The mode (λ_2, v_2) is unstable, so it will eventually blow up.

This mode is, however, not observable through the output.

The system is LTI ($q(0) = 0$), so outputs are sum of shifted $h(n)$ that we found in (b). Since $h(n)$ does not depend on mode (λ_2, v_2), all outputs will not depend on this mode.

MT2.5 (12 Points) Consider a nonlinear, continuous-time autonomous system whose state-evolution obeys the following:

$$\begin{cases} \dot{x} = x^2 - 1 \\ \dot{y} = -x\,y\,. \end{cases}$$

2 (a) Determine all the fixed points of the system. *Hint:* Recall that at a fixed point both of the time derivatives $\dot{x}$ and $\dot{y}$ must be zero. Find all such points in the xy-plane.

$$\left.\begin{array}{lll} \dot{x} = x^2 - 1 = 0 & \Rightarrow & x = \pm 1 \\ \dot{y} = -xy = 0 & \Rightarrow & x = 0 \text{ or } y = 0 \end{array}\right\} \Rightarrow y = 0,\ x = \pm 1$$

Fixed points are $(1, 0)$ and $(-1, 0)$

10 (b) Provide a well-labeled sketch of the phase portrait of the system, and classify the fixed points in terms of their stability. *Hint:* Find the x- and y-*nullclines* of the system. The x-nullcline is the set of points where $\dot{x} = 0$; there is no x-variation at any point along the x-nullcline. Similarly, the y-nullcline is the set of points where $\dot{y} = 0$; there is no y variation at any point along the y-nullcline. Identifying the nullclines makes sketching the vector field straightforward.

x-nullcline at $x = \pm 1$ $\Rightarrow$ vertical lines at $x = \pm 1$

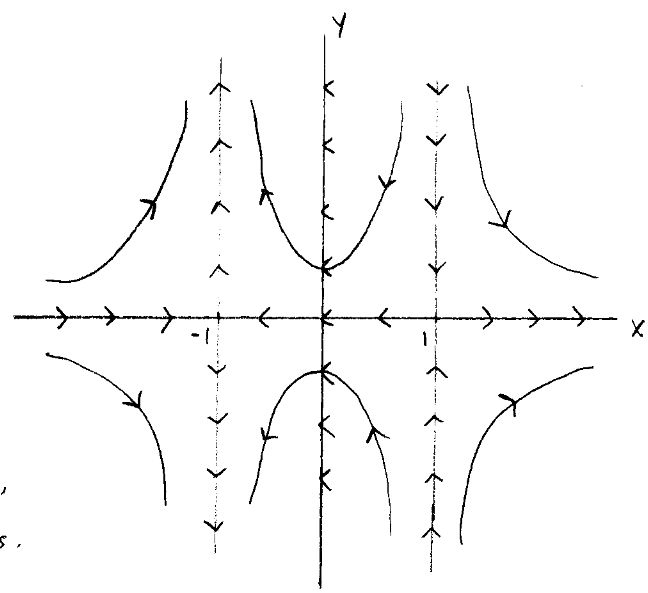
Look at $\dot{y} = -xy$ to find the vertical direction

y-nullcline at $x = 0$ or $y = 0$, i.e. y-axis and x-axis

Look at $x^2 - 1$ to find the horizontal direction

$\leftarrow$ for $x \in (-1, 1)$

$\rightarrow$ otherwise



By inspecting the phase portrait around the fixed points, we can see that both fixed points are saddle points.

LAST Name ____________ FIRST Name Solution ____________

Lab Time ____________

Problem	Points	Your Score
Name	10	10
1	15	15
2	30	30
3	30	30
4	18	18
5	12	12
Total	**115**	115