Grading for MT1.1A

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We are given the identity

$$\delta(\alpha f) = \frac{1}{|\alpha|}\delta(f) \tag{1}$$

And want to show that

$$\delta(\alpha(f - f_0)) = \frac{1}{|\alpha|}\delta(f - f_0) \tag{2}$$

There are three different ways to solve this problem:

1. Variable substitution
2. Integration (this is the approach you guys took in your homework to prove (1).)
3. Writing the δ function as the limit of box functions

We'll prove the identity using each of these techniques and discuss possible pitfalls and how the question was graded if you made one of these mistakes.

0.1 The box function

We first make the useful definition of the box function

$$\text{box}(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

This definition comes in handy for the last two proof techniques.

1 Variable substitution

Since (1) is true, we also must have that for some other continuous variable τ

$$\delta(\alpha\tau) = \frac{1}{|\alpha|}\delta(\tau) \tag{4}$$

Why are we allowed to just make up a new variable τ and write this relationship? Well, if we have a function g with domain $\mathbb{R}$ and range $\mathbb{R}$, it doesn't matter whether we use x to denote elements of the domain or the variable y to denote elements of the domain. Both are essentially placeholders.

As another example, we know that $g(x) = x^2$, $g(y) = y^2$ and $h(z) = z^2$ are all the same function (in this case, the square function.)

Anyway, with this complete, we are nearly done. Set

$$\tau = f - f_0. \tag{5}$$

This step is perfectly kosher mathematically. I could set $\tau = 1, 15.2, -7, x$, or whatever else I like so long as what I'm setting it to is an element of the function's domain (in this case, $\mathbb{R}$.)

With (4) and (5), we have all we need to solve this problem.

We then have the identities:

$$\begin{aligned}
\delta(\alpha(f - f_0)) &= \delta(\alpha\tau) \text{ by (5)} \\
&= \frac{1}{|\alpha|}\delta(\tau) \text{ by (4)} \\
&= \frac{1}{|\alpha|}\delta(f - f_0) \text{ again by (5)}
\end{aligned}$$

Which is exactly the result we wanted.

1.1 Commentary on grading

Variable substitution is the easiest way to solve the problem. However, because it is so easy, there isn't much room for partial credit if you make a mistake or don't explain yourself properly.

Here is how I graded this problem:

1. If I felt satisfied that you understood what was going on and you wrote up a correct proof similar to the one above, then you got the full 10 points.

2. If you made a small mistake of some sort but nailed the variable substitution, you got 5 points.

3. If you just wrote down the answer without any sort of justification along the lines of variable substitution, then you got 3 points. By far the biggest mistake people made was statements like "Set $f = f - f_0$." I *get* what you were trying to do, but this makes no mathematical sense, and doesn't convince me that you understand the two ingredients of the proof provided above.

4. If you made a major mistake of some sort that was pretty far from correct or wrote down absolutely nothing, you got either 0 or 1 points. There was a bit of discretion on my part here, but overall if I felt you tried to do something, I gave you the point.

2 Integration

To prove (2), we will integrate both $h_1(f) := \delta(\alpha(f - f_0))$ and $h_2(f) := \frac{1}{|\alpha|}\delta(f - f_0)$ against an arbitarily given continuous function $g(f)$ and show that the two integrals are equal. Since the test function g is arbitarily chosen, doing this is equivalent to showing that h_1 and h_2 act upon *all* continuous functions in the same way.

Which for the purposes of this class shows that h_1 and h_2 are equal. [1]

Intuitively, it makes sense that if two functions h_1 and h_2 satisfy this property that they must be equal.

Recall the definition of the box function $\mathrm{box}(x)$ from equation (3). We could always define a sequences of test functions $g_{m,n}$, $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}$, where $q_{m,n}$ is defined as

$$q_{m,n}(x) = \mathrm{box}(2^m[x - n])$$

This sequence of test functions gives us very fine control. If two functions are NOT equal, then intuitively we should be able to find a function in this class which we can integrate against and get two different numbers; our sequences of functions allows us to test every snippet of the domain for equality.

And hopefully it should be clear why we cannot just integrate against a SINGLE function $g(f)$! If we know that $\int g(f)h_1(f)df = \int g(f)h_2(f)df$, we don't have enough information to conclude that $h_1 = h_2$.

Let us first multiply and integrate against the LHS:

$$\begin{aligned}\int_{-\infty}^{\infty} g(f)\delta(\alpha[f - f_0])df &= \int_{-\infty}^{\infty} g(\tau/\alpha)\delta(\tau - \alpha f_0)\frac{d\tau}{|\alpha|} \text{ (After we set } \tau = \alpha f) \\ &= \frac{1}{|\alpha|}g(\frac{\alpha f_0}{\alpha}) \text{ (by basic properties of the } \delta \text{ function).}\end{aligned}$$

Processing the RHS, we get

$$\int_{-\infty}^{\infty} g(f)\frac{1}{|\alpha|}\delta(f - f_0)df = \frac{1}{|\alpha|}g(f_0) \text{ (by basic properties of the } \delta \text{ function).}$$

Since integrating against the LHS and the RHS gives the same result, then the two functions must be equal.

2.1 Commentary on grading

1. This proof technique is a bit harder than the variable substitition technique, so I graded those who tried it a bit more leniently.

2. 10 points if you provide a proof similar to the above.

[1] Notice that I punted here. Basically, there are technical details involved in defining this notion of equality, details beyond the scope of this class. If you are curious and would like to learn more, the Wikipedia article on the Dirac delta and the references it provides are a good place to start.

3. 7 points if you tried a proof similar to the above, but made a minor error.
4. 3 points for trying the above and making a significant error.
5. If you integrate only against the function $g(f) = 1$ but do this correctly, 5 points.

3 Approximating the δ function

We can define

$$q_\Delta(x) = \frac{1}{\Delta}\text{box}(\frac{1}{\Delta}[x - \frac{1}{2}])$$

We are allowed to interpret the δ function as

$$\delta(f) = \lim_{\Delta \to 0} q_\Delta(x). \tag{6}$$

This suggests that

$$\frac{1}{|\alpha|}\delta(f - f_0) = \frac{1}{|\alpha|}\lim_{\Delta \to 0} q_\Delta(\alpha[f - f_0]).$$

Let us look more closely at $\frac{1}{|\alpha|}q_\Delta(f - f_0)$. We have that

$$\begin{aligned}
\frac{1}{|\alpha|}q_\Delta(f - f_0) &= \frac{1}{|\alpha|}\begin{cases} \frac{1}{\Delta} & \text{if } |f - f_0| \leq \frac{\Delta}{2}, \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \frac{1}{\Delta|\alpha|} & \text{if } |\alpha(f - f_0)| \leq \frac{\Delta|\alpha|}{2}, \\ 0 & \text{otherwise.} \end{cases} \\
&= q_{\Delta|\alpha|}(\alpha[f - f_0]).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{|\alpha|}\delta(f - f_0) &= \frac{1}{|\alpha|}\lim_{\Delta \to 0} q_\Delta(\alpha[f - f_0]) \\
&= \lim_{\Delta \to 0} q_{\Delta|\alpha|}(\alpha[f - f_0]) \\
&= \lim_{\Delta|\alpha| \to 0} q_{\Delta|\alpha|}(\alpha[f - f_0]) \text{ (since } |\alpha| \text{ is a fixed constant.)} \\
&= \delta(\alpha[f - f_0]) \text{ (by (6).)}
\end{aligned}$$

Which is exactly what we needed to show.

3.1 Commentary on grading

Here is how I graded this problem:

1. If you provided a correct proof similar to that above, I gave you 10 points.
2. If you made a small mistake of some sort but had most of it, i gave you 7 points.
3. If you just drew a picture and gave a bad explanation, I gave you 1 point.

1 Problem 1.b

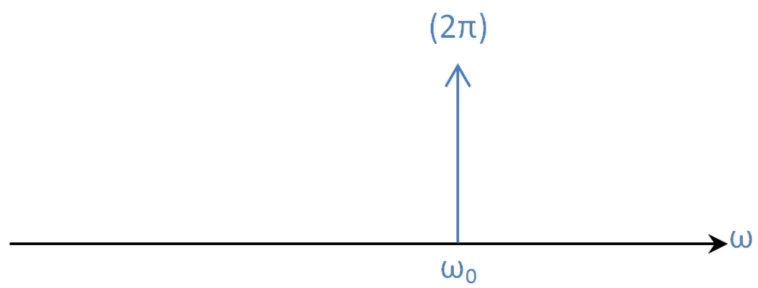
(6pts) We want to express $\delta(f - f_0)$ in terms of ω. Given that $\omega = 2\pi f$ and $\omega_0 = 2\pi f_0$:

$$f = \frac{1}{2\pi}\omega$$

$$f_0 = \frac{1}{2\pi}\omega_0$$

$$\delta\left(f - f_0\right) = \delta\left(\frac{1}{2\pi}\omega - \frac{1}{2\pi}\omega_0\right) = \delta\left(\frac{1}{2\pi}(\omega - \omega_0)\right) = \frac{1}{\left|\frac{1}{2\pi}\right|}\delta\left(\omega - \omega_0\right) = 2\pi\delta\left(\omega - \omega_0\right)$$

(4pt) Now, we want to plot this frequency spectrum. Since we have $2\pi\delta\left(\omega - \omega_0\right)$, the plot is simply a Dirac delta that is active at ω_0, with a strength of 2π:



Note: If you did not express $\delta(f - f_0)$ in terms of ω correctly, I did my best to not take away points if the plot correctly reflects the expression you got. For example, $\frac{1}{2\pi}\delta\left(\omega - \omega_0\right)$ was a common answer, and if the plot were such that the strength is labeled as $\frac{1}{2\pi}$, the plot would have not lost points for that.

Problem 2a:

If $b^2 - 4ac \geq 0$, then both roots are real.

If $b^2 - 4ac < 0$, then using the formula given in the problem, we see that

$$\begin{cases} z_1 = \frac{-b}{2a} + i\frac{\sqrt{4ac-b^2}}{2a} \\ z_2 = \frac{-b}{2a} - i\frac{\sqrt{4ac-b^2}}{2a} \end{cases}$$

And z_1 and z_2 are indeed conjugates of each other.

Remarks on some common errors:

- Some students forgot to change the radical to $\sqrt{4ac - b^2}$ when writing z_1 and z_2 (when z_1 and z_2 are complex); in this case, 1 point was taken off.
- Proving the result for only a certain set of a, b and c gets 0 for this problem.
- "$b^2 < 4ac$ implies $b < 2\sqrt{ac}$" is only true when $b \geq 0, ac \geq 0$. In this problem, this turns out to be true, but this proposition might not be true in certain cases, so when making such statements, you should be careful with the signs (of the students making this statement, only one student pays attention to this). However, no points were taken off for this.

There are a few ways to solve this equality and the following two are among those. If you have put another solution on the midterm but have gotten to the result correctly would have been given full credit.

We know from the question that $Z_1 + Z_2 = -\frac{b}{a}$ and $Z_1 * Z_2 = \frac{c}{a}$.

The hint says: Eliminate the coefficient a by combining the expressions for $Z_1 + Z_2$ and $Z_1 * Z_2$ in just right way. Since we see $\frac{c}{b}$ in the equation we want to prove, we figure that by dividing $Z_1 * Z_2$ by $-(Z_1 + Z_2)$ we can get $\frac{c}{b}$. So We replace $\frac{c}{b}$ in $|Z_l + \frac{c}{b}|$ with $\frac{Z1*Z2}{-(Z1+Z2)}$ which gives us

$$\left| Z_l - \frac{Z1*Z2}{(Z1+Z2)} \right|$$

If Z_l is Z_1 ➔ $\left| Z_1 - \frac{Z1*Z2}{(Z1+Z2)} \right| = \left| \frac{Z1*Z1+Z1*Z2-Z1*Z2}{(Z1+Z2)} \right| = \left| \frac{Z1*Z1}{(Z1+Z2)} \right| = \frac{|Z1*Z1|}{|Z1+Z2|}$

$= \frac{|Z1|\text{^}2}{|Z1+Z2|} = \frac{|Z1*Z2|}{|Z1+Z2|} = \frac{\frac{c}{a}}{\frac{b}{a}} = \frac{c}{b}$ $\qquad |z_1|^2 = |z_1||z_1| = |z_1||z_1^*| = |z_1||z_2| = |z_1 z_2|$

For $Z_l = Z_2$, the solution is the same.

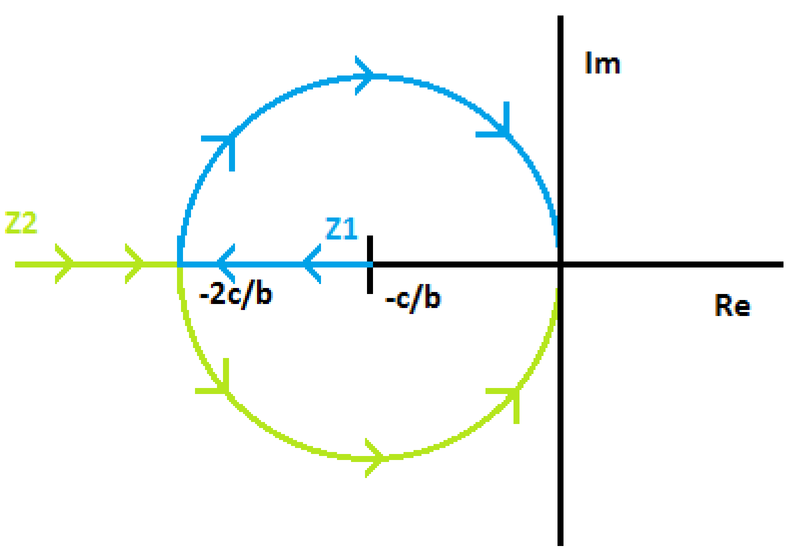
Ranging a from 0^+ to $+\infty$. First we identify when the roots leave the real axis. From the $\sqrt{b^2-4ac}$ term in both Z_1 and Z_2, we know that transition happens when $b^2-4ac=0$. Or equivalently, when $a=\frac{b^2}{4c}$. On the graph, this is a point at $Z_1=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{-b+\sqrt{0}}{2(\frac{b^2}{4c})}=\frac{-2c}{b}$. Note that for $a=\frac{b^2}{4c}$, Z_2 is also $\frac{-2c}{b}$, so this is a double root.

Now there are two ranges to consider: $0<a<\frac{b^2}{4c}$, for which Z_1 and Z_2 are real, and $\frac{b^2}{4c}<a<+\infty$, for which Z_1 and Z_2 are complex.

For $\frac{b^2}{4c}<a<+\infty$, as $a\to+\infty$ we have $Re\{Z_l\}=\frac{-b}{2a}\to 0$ and

$Im\{Z_l\}=\pm\frac{\sqrt{b^2-4ac}}{2a}\to 0$. In part (i) we determined that, for this range of a, $|Z_l-\frac{c}{b}|=\frac{c}{b}$. Therefore Z_1 and Z_2 approach 0 on a circle centered at $\frac{-c}{b}$, with radius $\frac{c}{b}$, where Z_1 travels along the top half of the circle and Z_2 along the bottom.

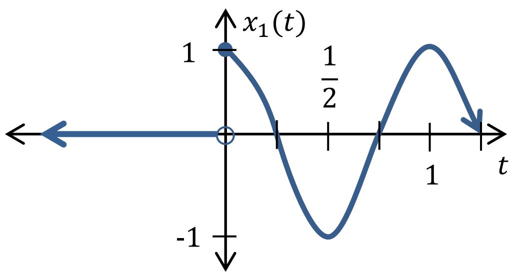
For $0<a<\frac{b^2}{4c}$, both Z_1 and Z_2 are real-valued and therefore plotted only on the real axis. They are no longer complex conjugates for this range of a. As $a\to 0^+$, $Z_1=\frac{-b+\sqrt{b^2-4ac}}{2a}\to\frac{-b+\sqrt{b^2}}{2(0)}\to\frac{0}{0}$. That is less than useful, so we use L'Hôpital's rule to find $Z_1=\frac{-b+\sqrt{b^2-4ac}}{2a}\to\frac{\frac{1}{2}(-4c)(b^2-4ac)^{\frac{-1}{2}}}{2}\to\frac{-2c}{2\sqrt{b^2}}\to\frac{-c}{b}$ as $a\to 0^+$. Z_2 is more easily found: as $a\to 0^+$, $Z_2=\frac{-b-\sqrt{b^2-4ac}}{2a}\to\frac{-b-\sqrt{b^2}}{2(0^+)}\to-\infty$. Therefore we draw Z_1 and Z_2 starting at $\frac{-c}{b}$ and $-\infty$, respectively, and moving along the real axis until they reach $\frac{-2c}{b}$ and split up again.

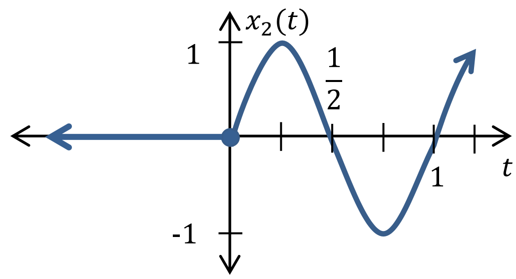


MT1.3 (40 Points) Parts (a) and (b) of this problem are independent, so you may approach them in either order.

(a) (25 Points) Consider the continuous-time signals x_1 and x_2 described by $x_1(t) = \cos(2\pi t)\, u(t)$ and $x_2(t) = \sin(2\pi t)\, u(t)$, for all t. The function u is the continuous-time unit step. These signals are called a *tone bursts*, in part because they represent suddenly-applied sinusoids.

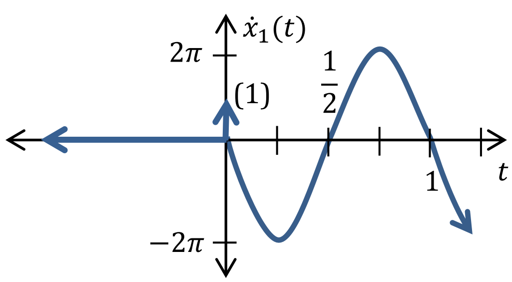
(i) (10 Points) Provide well-labeled plots of the signals x_1 and x_2. Explain how you label the important features of your plots.

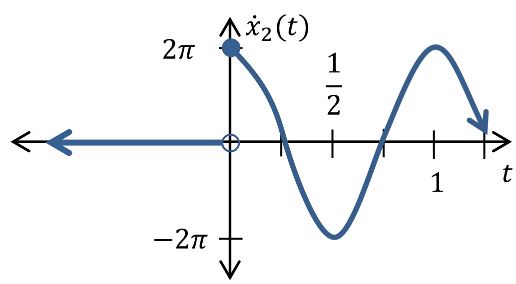




Note the open circle notation at $t = 0$ for x_1 at (0,0) while a closed circle notation at (0, 1).

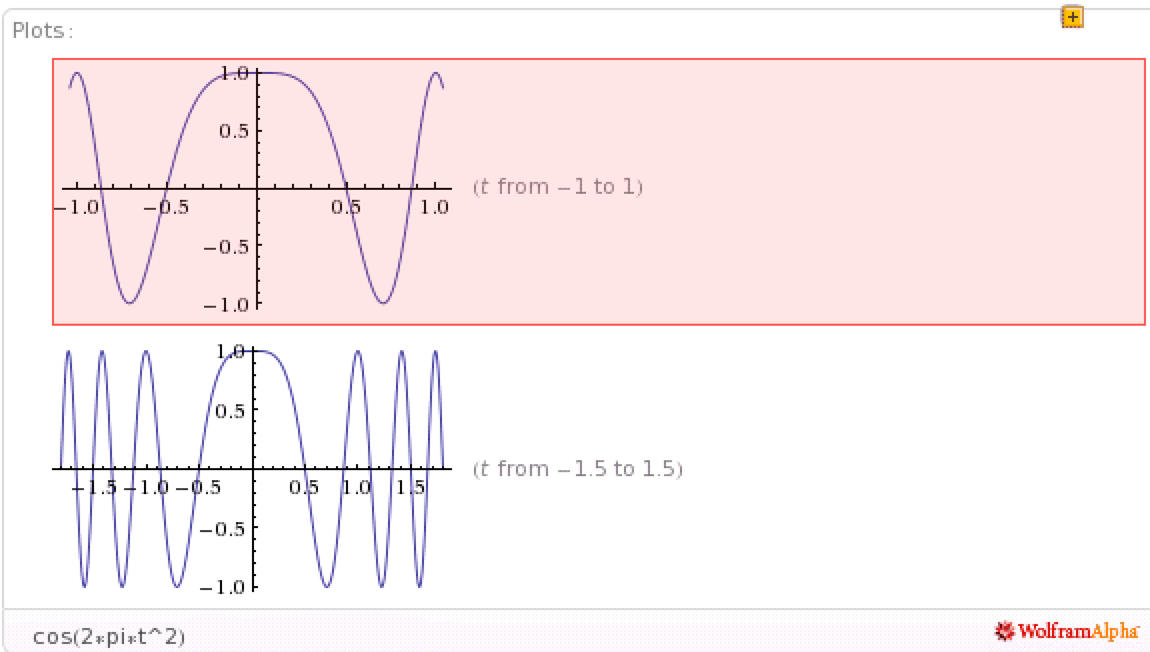
(ii) (15 Points) Determine, and provide well-labeled plots of, $\dot{x_1}(t)$ and $\dot{x_2}(t)$, the first-order time derivatives of the signals x_1 and x_2 .





Note that there is a Dirac Delta at $t = 0$ for $\dot{x_1}(t)$ with height 1. Also the curves have amplitude of 2π due to chain rule. Example: $\frac{d}{dt}\sin(\alpha t) = \alpha\cos(\alpha t)$ and in this case $\alpha = 2\pi$.

(b) (15 Point) Consider the continuous-time signal r described by $r(t) = \cos(\theta(t))$, where $\theta(t) = 2\pi t^2$, for all t. Provide a well-labeled plot of $r(t)$, for all t. Recall that the instantaneous frequency ω is given by $\omega(t) = \dot{\theta}(t) \triangleq \frac{d\theta(t)}{dt}$, for all t, so be sure to explain how the instantaneous frequency influences your plot.



Please be sure when making well-labeled plots to actually have labels! Also due to the nature of this graph such that the frequency increases or period decreases as magnitude of t increase, having just between -1 and 1 was not enough for full credit.

$$w(t) = \frac{d}{dt} 2\pi t^2 = 4\pi t$$

We can see from this that the instantaneous frequency is not constant! It is linearly dependent on t. Due to this relation we thus know that as the magnitude of t increases the magnitude of the frequency should increase as well and thus we should see the periods shortening in our graph as the magnitude of t increases.