EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 1
Department of Electrical Engineering and Computer Sciences 23 September 2008
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Phasor FIRST Name Oscilla

Lab Time Puleeez!

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" × 11" sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT1.1 (20 Points) Consider the sets

$$A = \{1, 2, 3\} \qquad \text{and} \qquad B = \{a, b\}.$$

For each part, explain your reasoning succinctly, but clearly and convincingly.

(a) Determine the set $A \times B$ explicitly.

$A \times B = \{(\alpha, \beta) \mid \alpha \in A \wedge \beta \in B\}$ $\qquad |A \times B| = |A||B|$

$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

As expected, $|A \times B| = 6$

(b) Determine $|P(A \times B)|$, the cardinality of the power set of $A \times B$. You should be able to reason this out *without* listing the elements of the power set $P(A \times B)$.

$|A \times B| = 6 \implies |P(A \times B)| = 2^{|A \times B|} = 2^6 = 64$

For a generic set Γ of size N, $|P(\Gamma)| = 2^N$, because in looking at each element γ in Γ, we must make a binary decision as to whether to include it as an element G of $P(\Gamma)$, where $G \subset \Gamma$.

(c) How many functions $f : A \to B$ can we define?

Every function $f : A \to B$ must map each element in A to exactly one element in B. For each element in A, there are two choices in B. Hence, there are $2^3 = 8$ functions in $[A \to B]$, i.e., $\boxed{|[A \to B]| = 8}$

An example is described as $graph(f) = \{(1,a), (2,a), (3,b)\}$.

(d) How many *onto* functions $f : A \to B$ can we define?

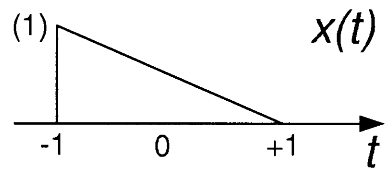
Only two functions in $[A \to B]$ are not onto. They are described by $graph(f_1) = \{(1,a), (2,a), (3,a)\}$ and $graph(f_2) = \{(1,b), (2,b), (3,b)\}$.

The remaining six functions in $[A \to B]$ are onto.

(e) How many *one-to-one* functions $f : A \to B$ can we define?

None. There aren't enough elements in B to allow for a one-to-one function to exist in $[A \to B]$. The set B must have at least three elements for an unto function to exist.

MT1.2 (30 Points) A continuous-time signal x is shown in the figure below. The signal is zero outside the interval shown.

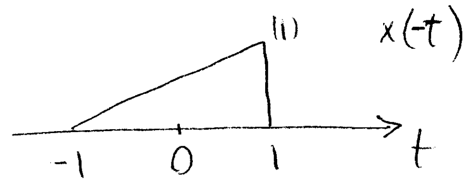


(a) Provide well-labeled sketches of $x_e(t)$ and $x_o(t)$, which correspond to the even and odd parts of x, respectively.

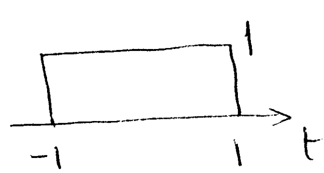
Recall that

$$x_e(t) = \frac{x(t) + x(-t)}{2} \qquad \text{and} \qquad x_o(t) = \frac{x(t) - x(-t)}{2}.$$

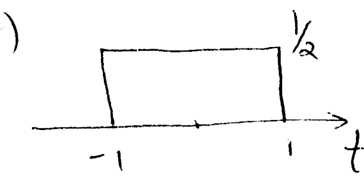
The time-reversed version of x is

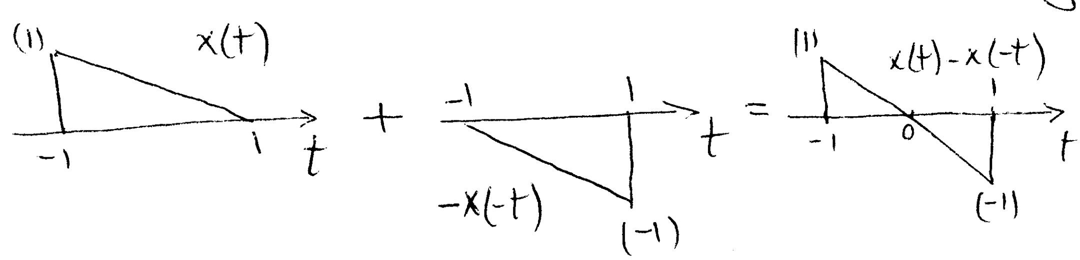


x(t) + x(-t)

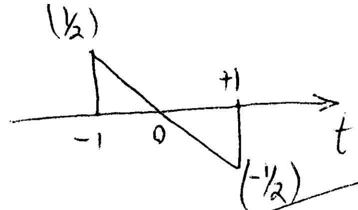


$\Rightarrow$ $x_e(t)$



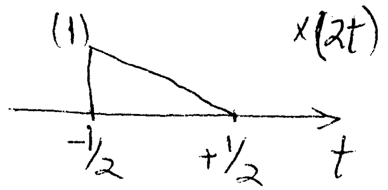


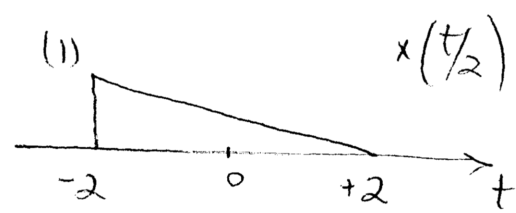
$x_o(t)$



(b) Provide well-labeled sketches of $x(2t)$ and $x(t/2)$.

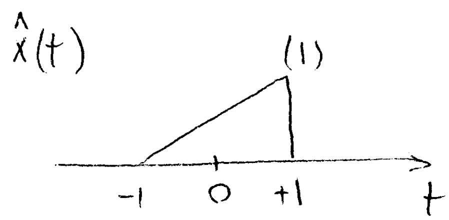
$x(2t)$ is a time-compressed version of $x(t)$ (by a factor of 2), whereas $x(t/2)$ is a time-dilated version of $x(t)$ (by a factor of 2). So the plots are as shown below:



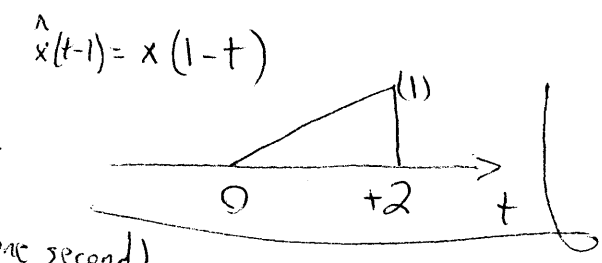


(c) Provide well-labeled sketches of $x(1-t)$ and $x(-1-t)$.

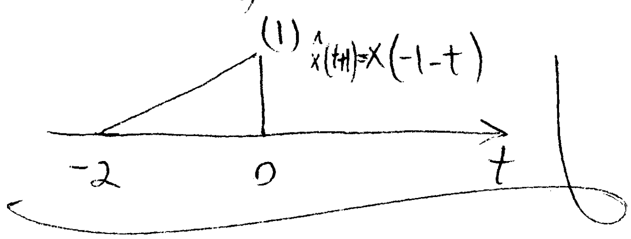
Let $\hat{x}(t) = x(-t)$ be the time-reversed version of x. Then $\hat{x}(t-1) = x(-(t-1)) = x(1-t)$ and $\hat{x}(t+1) = x(-(t+1)) = x(-1-t)$.



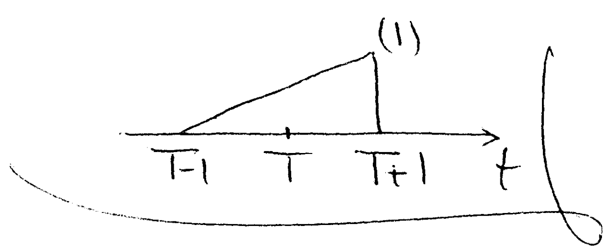
shift to the right (delay by one second)



Shift to the left (advance by one second)

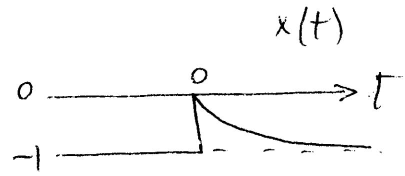


In general, $x(T-t)$



MT1.3 (20 Points) A continuous-time signal x described by

$$\forall t \in \mathbb{R}, \quad x(t) = \begin{cases} -1 & t < 0 \\ e^{-t} - 1 & t \geq 0. \end{cases}$$

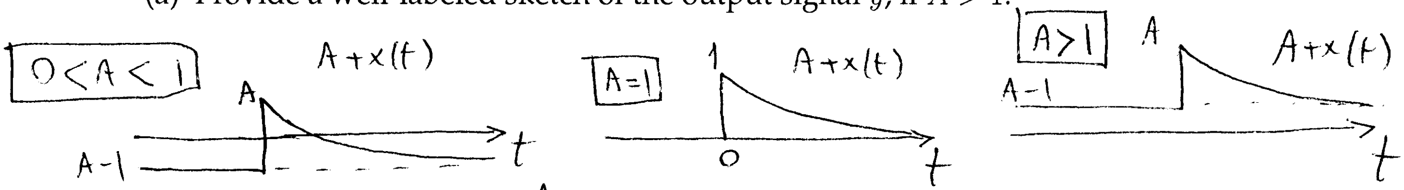


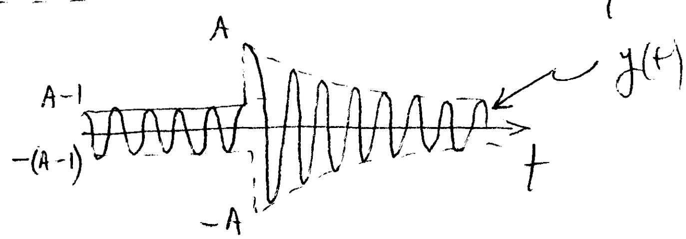
The signal x is sent through a special amplitude modulator. The modulator's output signal y is

$$\forall t \in \mathbb{R}, \quad y(t) = [A + x(t)]\cos(\omega_0 t),$$

where $\omega_0 \gg 1$ and $A > 0$.

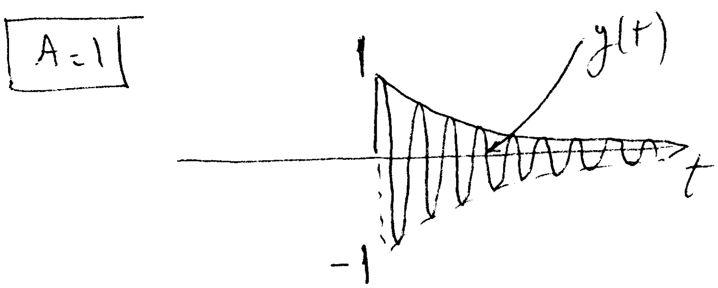
(a) Provide a well-labeled sketch of the output signal y, if $A > 1$.





y(t) peaks at each side of t=0.

(b) Provide a well-labeled sketch of the output signal y, if $A = 1$.



MT1.4 (35 Points) A continuous-time signal x described by

$$\forall t \in \mathbb{R}, \quad x(t) = \frac{1}{4} e^{i(\omega_0 + \omega_1)t} + e^{i\omega_0 t} + \frac{1}{4} e^{i(\omega_0 - \omega_1)t},$$

represents the trajectory of a particle on the complex plane. We may think of $x(t)$ as the instantaneous position of the particle at time t.

Throughout this problem, assume that the frequencies ω_0 and ω_1 are strictly positive (i.e., $\omega_0 > 0$ and $\omega_1 > 0$).

(a) Show that $x(t)$ can be written in the polar form

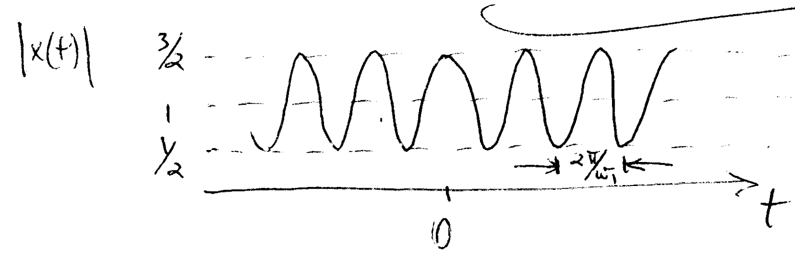
$$x(t) = |x(t)| e^{i\angle x(t)}.$$

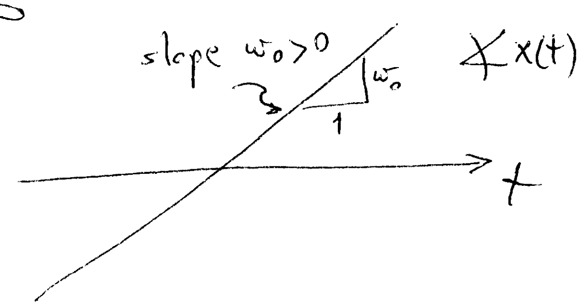
Determine explicit and reasonably simple expressions for the instantaneous magnitude $|x(t)|$ and the instantaneous phase $\angle x(t)$ of the particle's position, and provide a well-labeled a sketch of each.

$$x(t) = \left(\frac{1}{4} e^{i\omega_1 t} + 1 + \frac{1}{4} e^{-i\omega_1 t} \right) e^{i\omega_0 t} = \left[1 + \frac{1}{2}\cos(\omega_1 t) \right] e^{i\omega_0 t}$$

Note that $1 + \frac{1}{2}\cos(\omega_1 t) \geq \frac{1}{2} > 0$, $\forall t \Rightarrow |x(t)| = 1 + \frac{1}{2}\cos(\omega_1 t)$

The phase is $\angle x(t) = \omega_0 t$





(b) For this part, let $\omega_1 = 4\omega_0$.

(i) Provide a well-labeled sketch of the trajectory of the particle as it moves on the complex plane with the *forward* passage of time (in particular, as $0 \to t \to \infty$).
Note: Do *not* sketch $\mathrm{Re}(x(t))$, $\mathrm{Im}(x(t))$, $|x(t)|$, or $\angle x(t)$ in this part. You may receive credit only if you sketch $x(t)$, $0 \le t < \infty$, as a trajectory on the complex plane.

(ii) Denote, with one or more arrows, the direction of the particle's forward travel in time along the trajectory.

(iii) Determine the exact location of the particle at the first time instant $t_0 > 0$ where $\angle x(t_0) = \pi/4$. Identify the location $x(t_0)$ on the trajectory sketch that you provided in Part (b)(i).

$$\left.\begin{array}{l} x(t) = \left[1 + \frac{1}{2}\cos(\omega_1 t)\right] e^{i\omega_0 t} \\ \omega_1 = 4\omega_0 \end{array}\right\} \Rightarrow x(t) = \left[1 + \frac{1}{2}\cos(4\omega_0 t)\right] e^{i\omega_0 t}$$

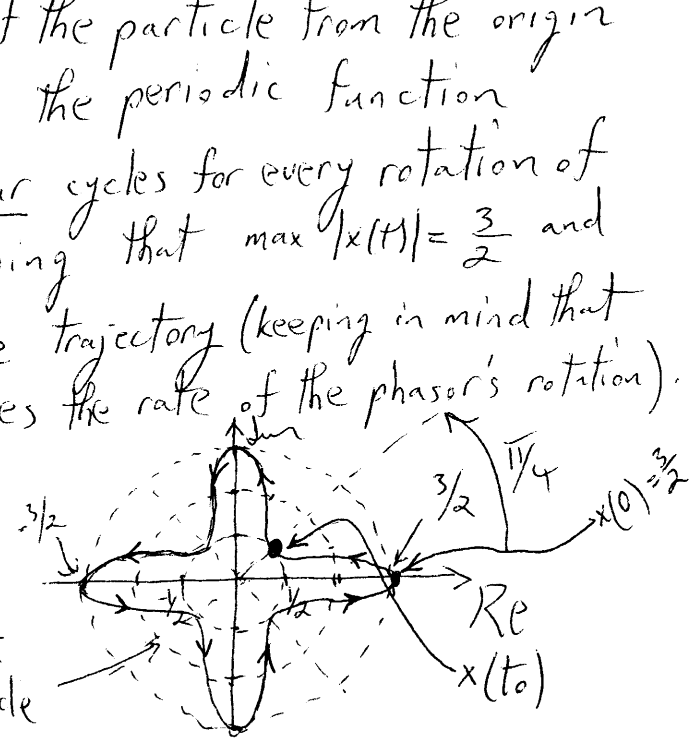
The phasor $e^{i\omega_0 t}$, if left to its own devices, rotates counter-clockwise at angular frequency ω_0 rads/sec on the unit circle. However, the magnitude $|x(t)| = 1 + \frac{1}{2}\cos(4\omega_0 t)$ prevents the phasor $e^{i\omega_0 t}$ from staying on the unit circle. In fact, $|x(t)|$ modulates the length of the phasor (i.e., the radial distance of the particle from the origin on the complex plane) according to the periodic function $1 + \frac{1}{2}\cos(4\omega_0 t)$, which completes four cycles for every rotation of the phasor around the origin. Knowing that $\max |x(t)| = \frac{3}{2}$ and $\min |x(t)| = \frac{1}{2}$, we can now draw the trajectory (keeping in mind that the magnitude oscillates at exactly four times the rate of the phasor's rotation).

$$\angle x(t_0) = \pi/4 \Rightarrow \omega_0 t_0 = \frac{\pi}{4} \Rightarrow t_0 = \frac{\pi}{4\omega_0} = \frac{\pi}{\omega_1}$$

But π/ω_1 is half the period of $|x(t)|$,

so it is at its minimum: $|x(t_0)| = \left|1 + \frac{1}{2}\cos\left(4\omega_0 \frac{\pi}{4\omega_0}\right)\right| = \frac{1}{2}$

$x(t_0) = \frac{1}{2} e^{i\pi/4}$



LAST Name Phasor FIRST Name Oscilla
Lab Time Puleeez!

Problem	Points	Your Score
Name	10	10
1	20	20
2	30	30
3	20	20
4	35	35
Total	**115**	115