EE 20

Midterm 1

October 1, 2003

<u>Solutions</u>

Note: For some of the problems, there are many possible answers.

In these cases, an example solution is provided.

Name: ______________________________

Problem 1: 10 Points Possible

Use mathematical notation to define the following sets.

a) The set of all integers which are divisible by 4 (with zero remainder).

$\{x \in \text{Integers} \mid x / 4 \in \text{Integers}\}$

b) The set of all natural numbers which are also prime numbers.

$\{x \in \text{Naturals} \setminus \{1\} \mid \forall\, q \in \text{Naturals} \setminus \{1, x\},\ x / q \notin \text{Naturals}\}$

Name: ______________________________

Problem 2: 15 Points Possible

Give an example of a system H that takes functions in the specified domain as input, and produces functions in the specified domain as output. Provide a mathematical function definition of your system (that means tell me what your system H does to an input x, using mathematical notation). Your H(x) must depend on x (e.g., H(x)(t) = t not allowed.)

Example Question: $H: [\text{Naturals}_0 \longrightarrow \text{Reals}] \longrightarrow [\text{Reals}_+ \longrightarrow \text{Reals}]$

Example Possible Answer: $\forall\, x \in [\text{Naturals}_0 \longrightarrow \text{Reals}]$ and $\forall\, t \in \text{Reals}_+$

$$H(x)(t) = x(\lfloor t \rfloor)$$

a) $H: [\text{Reals} \longrightarrow \text{Reals}] \longrightarrow [\text{Naturals}_0 \longrightarrow \text{Reals}_+]$

$\forall\, x \in [\text{Reals} \longrightarrow \text{Reals}]$ and $\forall\, n \in \text{Naturals}_0$

$$H(x)(n) = |x(n)|$$

b) $H: [\text{Reals} \longrightarrow \text{Complex}] \longrightarrow [\text{Reals} \longrightarrow \text{Reals}^2]$

$\forall\, x \in [\text{Reals} \longrightarrow \text{Complex}]$ and $\forall\, t \in \text{Reals}$

$$H(x)(t) = (\text{Re}\{x(t)\}, \text{Im}\{x(t)\})$$

c) $H: [\text{Naturals}_0 \longrightarrow \text{Reals}] \longrightarrow [\text{Naturals}_0 \longrightarrow \{0, 1, 2, 3\}]$

$\forall\, x \in [\text{Naturals}_0 \longrightarrow \text{Reals}]$ and $\forall\, n \in \text{Naturals}_0$

$$H(x)(n) = \lfloor x(n) \rfloor \bmod 4$$

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Problem 3: 15 Points Possible

Consider the system ConvertBW.

ConvertBW takes a 1024 by 768 pixel, 24-bit color image as input.

The output of ConvertBW is a 1024 by 768 pixel black-and-white image. The color of a black-and-white image is represented by 1 bit: 0 for black and 1 for white.

For each pixel in the input image, ConvertBW sets the corresponding output pixel to white (1) if the average of the red, green, and blue color values of the input pixel is at least 128. If the average of the red, green, and blue color values of the input pixel is strictly less than 128, ConvertBW sets the output pixel to black (0).

Example: For an input image x, if x(1,1) = (128, 127, 126) then ConvertBW(x)(1,1) = 0 .

Give the domain, range, and mathematical function definition for the system ConvertBW. You need to specify, using mathematical notation, the output ConvertBW(x) for any input image x.

Domain = $[\{1, 2, 3, \ldots, 1024\} \times \{1, 2, 3, \ldots, 768\} \rightarrow \{1, 2, 3, \ldots, 2^8\}^3]$

Range = $[\{1, 2, 3, \ldots, 1024\} \times \{1, 2, 3, \ldots, 768\} \rightarrow \{0, 1\}]$

$\forall\, x \in$ Domain and $\forall\, (i, j) \in \{1, 2, 3, \ldots, 1024\} \times \{1, 2, 3, \ldots, 768\}$

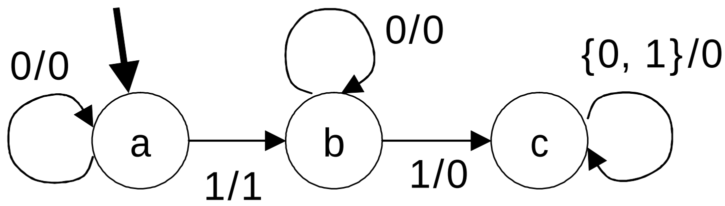
$$\mathrm{ConvertBW}(x)(i,j) = \begin{cases} 1 & \text{if } \dfrac{x_1(i,j) + x_2(i,j) + x_3(i,j)}{3} \geq 128 \\ 0 & \text{otherwise} \end{cases}$$

You may use any type of matrix/array notation you wish to specify the individual color values; For example, $x_3(1,1)$ and x(1,1)(3) are both fine ways to specify the blue value for pixel (1,1). You may treat x(i, j) as a column vector if you wish. Whatever you do, just explain it.

Name: ______________________________

Problem 4: 15 Points Possible

Consider the state machine below:



State Machine A

a) Write the 5-tuple for State Machine A. All the possible inputs and outputs are shown on the diagram. You may specify *update* using a table if you wish. Ignore the *absent* input, *absent* output, and stuttering reaction.

States = {a, b, c}

Inputs = {0, 1}

Outputs = {0, 1}

initialState = a

	(s(n+1), y(n))= update(s(n), x(n))	
Current State s(n)	Input x(n) = 0	Input x(n) = 1
a	(a, 0)	(b, 1)
b	(b, 0)	(c, 0)
c	(c, 0)	(c, 0)

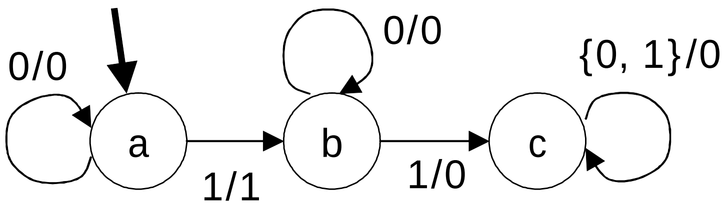
b) Consider an input sequence x which fits the pattern $x = 0^k\ 1^m\ *^n$ where $*$ can be a 0 or a 1. What is the corresponding output sequence y?

$y = 0^k\ 1\ 0^{m+n-1}$

Name: ______________________________

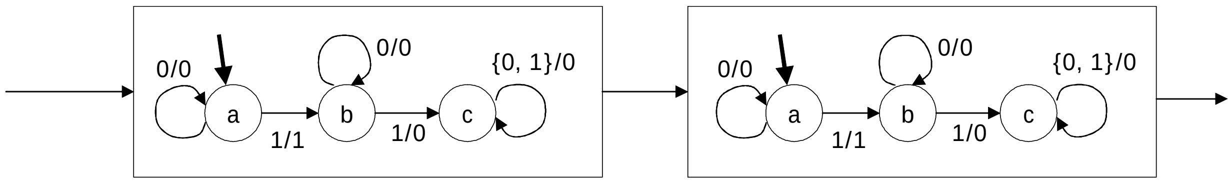
Problem 5: 15 Points Possible

Consider again the state machine



State Machine A

Cascade this state machine with itself:



List the unreachable states in this composition.

Hint: Although you are not required to give justification for your answer, to receive maximum partial credit if incorrect, you should show work or reasoning leading to your answer.

The machine starts out in (a,a).
It stays there until an input of 1 is received.
Then the machine goes to (b,b).
It stays there until another input of 1 is received.
Then the machine goes to (c, b).
It then stays there forever.
So the only reachable states are {(a, a), (b, b), (c, b)}.

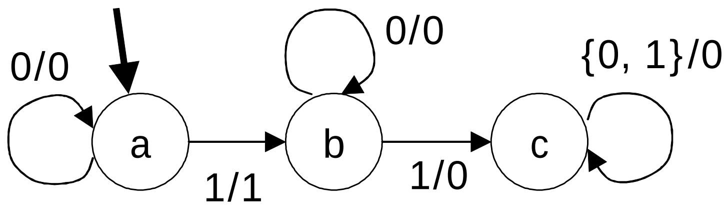
Hence, the unreachable states are:

{(a,b), (a, c), (b, a), (b, c), (c, a), (c, c)}

Name: ______________________________

Problem 6: 15 Points Possible

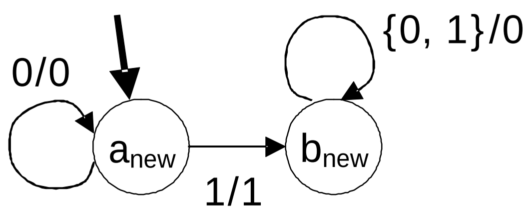
Consider again the state machine



State Machine A

Draw the diagram of a two-state machine, Machine B, that simulates Machine A. Give the simulation relation.

Your Machine B may be deterministic or nondeterministic; it is your choice.



Simulation relation: $\{(a, a_{new}), (b, b_{new}), (c, b_{new})\}$

Name: ______________________________

Problem 7: 15 Points Possible

Consider the feedback system, with *Inputs* = *Outputs* = $Reals_+$ and with the property

$y(n+1) = y(n)^p$ where p is a fixed value in Reals .

Determine the fixed points of this system, if they exist. Determine whether the fixed points are stable, and determine the domain of attraction for each stable fixed point.

These answers will be different for different values of p. Try to answer the question for as many values of p as you can. You will get full credit if you can provide the fixed points, stability, and domains of attraction for all values of $p \in$ Reals .

For $p > 1$:

Fixed points: {0, 1} 0 is stable with domain of attraction [0,1). 1 is unstable.

For $p = 1$:

Fixed points: $Reals_+$. Every fixed point is unstable.

For $p \in (0, 1)$:

Fixed points: {0, 1}. 1 is stable with domain of attraction $Reals_+ \setminus \{0\}$. 0 is unstable.

For $p = 0$:

Fixed points: {1}. 1 is stable with domain of attraction $Reals_+ \setminus \{0\}$.

For $p \in (-1, 0)$:

Fixed points: {1}. 1 is stable with domain of attraction $Reals_+ \setminus \{0\}$.

For $p = -1$:

Fixed points: {1}. 1 is unstable.

For $p < -1$:

Fixed points: {1}. 1 is unstable.