EE 20N: Structure and Interpretation of Signals and Systems
Department of Electrical Engineering and Computer Sciences — MIDTERM 1
UC BERKELEY — 23 September 2014

FIRST Name MIDTERM LAST Name SOLUTIONS

Lab Day & Time: ϕ SID (All Digits): ϕ

- **(10 Points)** Print your *official* name (not your e-mail address) and *all* digits of your student ID number legibly, and indicate your lab time, on *every* page.
- This exam should take up to 80 minutes to complete. You will be given at least 80 minutes, up to a maximum of 90 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

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MT1.1 (15 Points) The input to a discrete-time system S is described by

$$\forall n \in \mathbb{Z}, \qquad x(n) = \left(\frac{1}{2}\right)^n e^{i\pi n/4},$$

and the corresponding output is

$$\forall n \in \mathbb{Z}, \qquad y(n) = \left(\frac{1}{4}\right)^n e^{i\pi n/4}.$$

Select the strongest true assertion from the list below.

(i) The system S must be LTI.

(ii) The system S could be LTI, but does not have to be.

(iii) The system S cannot be LTI.

To receive credit, you must provide a succinct, but clear and convincing, explanation. Note that $x(n)$ above $= \left(\frac{1}{2}e^{i\pi/4}\right)^n$

<u>If S were LTI</u> (with some impulse response $s(n)$), then the output corresponding to input $x(n)$ should be able to be written in the form:

$$\sum_{k=-\infty}^{\infty} s(k)\,x(n-k) = \sum_{k=-\infty}^{\infty} s(k)\left(\tfrac{1}{2}e^{i\pi/4}\right)^{n-k}$$

$$= \underbrace{\left[\sum_{k=-\infty}^{\infty} s(k)\left(\tfrac{1}{2}e^{i\pi/4}\right)^{-k}\right]}_{\text{(just some complex number that doesn't depend on } n)} \cdot \left(\tfrac{1}{2}e^{i\pi/4}\right)^n$$

Looking at the $y(n)$ given in the problem, we can write $y(n)$ as: $y(n) = \left(\frac{1}{2}\right)^n \left(\frac{1}{2}e^{i\pi/4}\right)^n$, but since the "$\left(\frac{1}{2}\right)^n$" term depends on n, there is no way to write $y(n)$ in <u>this form</u>.

Therefore, the system cannot be LTI.

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MT1.2 (15 Points) The spectrum of a continuous-time 24kHz complex exponential signal q described by

$$\forall t \in \mathbb{R}, \qquad q(t) = e^{i48000\pi t},$$

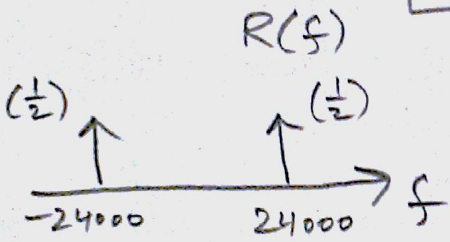
is given by $Q(f) = \delta(f - 24000)$, where f denotes frequency in Hertz, and δ is the Dirac impulse.

(a) (5 Points) Determine a reasonably-simple expression for, and provide a well-labeled plot of, the spectrum $R(f)$ of the signal r described by

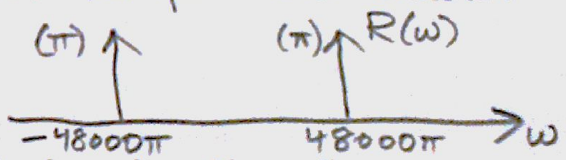
$$\forall t \in \mathbb{R}, \qquad r(t) = \cos(48000\pi t).$$

We are given that $q(t) = e^{i48000\pi t} = e^{i(24000)(2\pi)t}$ corresponds to $Q(f) = \delta(f-24000)$.

Since $r(t) = \cos(48000\pi t) = \frac{1}{2}e^{i(24000)(2\pi)t} + \frac{1}{2}e^{-i(24000)(2\pi)t}$, we have that $\boxed{R(f) = \frac{1}{2}\delta(f-24000) + \frac{1}{2}\delta(f+24000)}$



Plot for part (b) below:



(b) (10 Points) Determine a reasonably-simple expression for, and provide a well-labeled plot of, $R(\omega)$, the spectrum, as a function of the radial frequency ω (in radians/second), of the signal r. To receive full credit, you must provide a succinct, yet clear and convincing, explanation.

$\omega = 2\pi f$ or $f = \frac{\omega}{2\pi}$

$$\Rightarrow R(f) = \frac{1}{2}\delta(f-24000) + \frac{1}{2}\delta(f+24000) \qquad \text{(from (a))}$$

$$= \frac{1}{2}\delta\left(\frac{\omega}{2\pi} - 24000\right) + \frac{1}{2}\delta\left(\frac{\omega}{2\pi} + 24000\right)$$

$$= \frac{1}{2}\delta\left(\frac{\omega - 48000\pi}{2\pi}\right) + \frac{1}{2}\delta\left(\frac{\omega + 48000\pi}{2\pi}\right)$$

Recall the dilation/contraction property of the Dirac delta:

$$\delta(\alpha\omega) = \frac{1}{|\alpha|}\delta(\omega) \Rightarrow \delta(\alpha(\omega-\beta)) = \frac{1}{|\alpha|}\delta(\omega-\beta)$$

apply this to the deltas above $(\alpha = \frac{1}{2\pi}, \beta = 48000\pi$ or $-48000\pi)$ to get:

$$R(\omega) = \pi\delta(\omega - 48000\pi) + \pi\delta(\omega + 48000\pi)$$

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MT1.3 (75 Points) Consider a discrete-time filter F whose input is x and corresponding output is y, as shown in the figure below:

The output y of the filter is described by

$$\forall n \in \mathbb{Z}, \qquad y(n) = (1-\alpha)\sum_{m=0}^{\infty} \alpha^m x(n-m),$$

where $0 < \alpha < 1$.

(a) (10 Points) Show that the filter F is linear *and* time invariant.

Linearity:

Let x_1 and x_2 be any 2 input signals, and let y_1 and y_2 be their corresponding outputs.

Let $c_1, c_2 \in \mathbb{C}$.

Define $\hat{x}(n) \triangleq c_1 x_1(n) + c_2 x_2(n)$.

Let $\hat{y}(n)$ be the output corresponding to $\hat{x}(n)$.

$$\hat{y}(n) = (1-\alpha)\sum_{m=0}^{\infty} \alpha^m \hat{x}(n-m) = (1-\alpha)\sum_{m=0}^{\infty} \alpha^m [c_1 x_1(n) + c_2 x(n)]$$

$$= c_1 \underbrace{\left((1-\alpha)\sum_{m=0}^{\infty}\alpha^m x_1(n-m)\right)}_{=y_1(n)} + c_2 \underbrace{\left((1-\alpha)\sum_{m=0}^{\infty}\alpha^m x_2(n-m)\right)}_{=y_2(n)}$$

$$= c_1 y_1(n) + c_2 y_2(n) \quad \checkmark \quad \text{(system is linear)}$$

Time-invariance: Define $\tilde{x}(n) \triangleq x_1(n-N)$ for some $n \in \mathbb{Z}$

Let $\tilde{y}(n)$ be the corresponding output signal.

$$\tilde{y}(n) = (1-\alpha)\sum_{m=0}^{\infty}\alpha^m \tilde{x}(n-m)$$

$$= (1-\alpha)\sum_{m=0}^{\infty}\alpha^m x_1(n-N-m)$$

$$= y_1(n-N) \quad \checkmark \quad \text{(system is time-invariant)}$$

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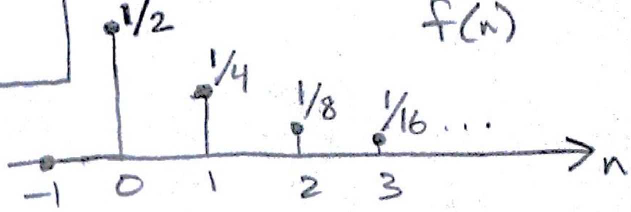
(b) (15 Points) Determine a reasonably-simple expression for, and provide a well-labeled plot of, the impulse response f of the filter.

Your expression for $f(n)$ must be in terms of the generic parameter α. However, for your convenience, use the value $\alpha = 1/2$ for your plot in this part.

$f(n)$ is the output when $\delta(n)$ (the Kronecker delta) is the input

$$\boxed{\begin{aligned} f(n) &= (1-\alpha)\sum_{m=0}^{\infty} \alpha^m \delta(n-m) \\ &= (1-\alpha)\,\alpha^n u(n) \end{aligned}}$$

if $\alpha = \frac{1}{2}$:



(c) (15 Points) Determine a reasonably-simple expression for (but do NOT attempt to plot) the filter's frequency response values $F(\omega)$ for all $\omega \in \mathbb{R}$. As a "sanity check," show that your expression for $F(\omega)$ is 2π-periodic in ω.

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-i\omega n} = \sum_{n=-\infty}^{\infty} \left((1-\alpha) \sum_{m=0}^{\infty} \alpha^m \delta(n-m) \right) e^{-i\omega n}$$

$$= (1-\alpha) \sum_{m=0}^{\infty} \alpha^m \underbrace{\sum_{n=-\infty}^{\infty} \delta(n-m) e^{-i\omega n}}_{= e^{-i\omega m}}$$

$$= (1-\alpha) \sum_{m=0}^{\infty} \alpha^m e^{-i\omega m} = (1-\alpha) \underbrace{\sum_{m=0}^{\infty} (\alpha e^{-i\omega})^m}$$

note that $|\alpha e^{-i\omega}| < 1$

use sum of a geometric series formula

$$\boxed{F(\omega) = (1-\alpha) \cdot \frac{1}{1-\alpha e^{-i\omega}}}$$

Sanity check:

$$F(\omega + 2\pi) = \frac{1-\alpha}{1-\alpha e^{-i(\omega+2\pi)}} = \frac{1-\alpha}{1-\alpha e^{-i\omega} e^{-i2\pi}} = \frac{1-\alpha}{1-\alpha e^{-i\omega}} \quad \left(\text{since } e^{-i2\pi} = 1\right)$$

$$= F(\omega) \quad \checkmark$$

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(d) (15 Points) For each of the following inputs x_i, where $i = 1, 2, 3$, determine a reasonably-simple expression for the corresponding output y_i. If you need, but are unsure of, your result from part (c) (depending on your approach, you may not need it), you may receive partial credit if you express your answers to this part in terms of the frequency response generically.

(i) (5 Points) $x_1(n) = 1$ for all integers n.

$$x_1(n) = e^{i0n} \Longrightarrow y_1(n) = F(0)e^{i0n} = F(0)$$

$$F(0) = \frac{1-\alpha}{1-\alpha e^{-i0}} = \frac{1-\alpha}{1-\alpha} = 1$$

$$\Longrightarrow \boxed{y_1(n) = 1}$$

(ii) (5 Points) $x_2(n) = \cos(\pi n)$ for all integers n.

$$x_2(n) = e^{i\pi n} \Longrightarrow y_2(n) = F(\pi)e^{i\pi n}$$

$$F(\pi) = \frac{1-\alpha}{1-\alpha e^{-i\pi}} = \frac{1-\alpha}{1+\alpha}$$

$$\Longrightarrow \boxed{y_2(n) = \frac{1-\alpha}{1+\alpha}\cos(\pi n) = \frac{1-\alpha}{1+\alpha}(-1)^n}$$

Note that
1) $e^{i\pi n}$
2) $\cos(\pi n)$
3) $(-1)^n$
are all the same

(iii) (5 Points) For all integers n,

$$x_3(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Observe that $x_3(n) = \frac{1}{2}(-1)^n + \frac{1}{2} = \frac{1}{2}x_2(n) + \frac{1}{2}x_1(n)$

By linearity, we know that

$$y_3(n) = \frac{1}{2}y_2(n) + \frac{1}{2}y_1(n)$$

$$\Longrightarrow \boxed{\begin{aligned} y_3(n) &= \frac{1}{2}\left(\frac{1-\alpha}{1+\alpha}(-1)^n\right) + \frac{1}{2}(1) \\ &= \begin{cases} \frac{1}{1+\alpha}, & \text{if } n \text{ is even} \\ \frac{\alpha}{1+\alpha} & \text{if } n \text{ is odd} \end{cases} \end{aligned}}$$

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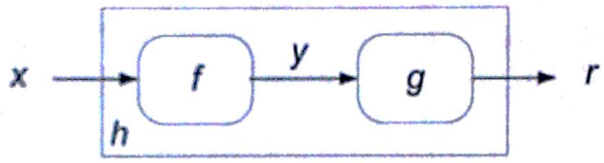
(e) (20 Points) The filter F is placed in a cascade (i.e., series) interconnection with a discrete-time LTI filter G whose impulse response g is described by

$$\forall n \in \mathbb{Z}, \qquad g(n) = \frac{1}{1-\alpha}\delta(n) - \frac{\alpha}{1-\alpha}\delta(n-1),$$

where α is the same as that used in the description of the filter F. Note that the output y of the filter F serves as the input to the filter G. This is shown in the figure below:



(i) (10 Points) Show that the system H, which consists of the cascade interconnection of F and G, is linear *and* time invariant.

We know that F is LTI (1.3(a)) and we are told that G is linear.

Let $x_1(n), x_2(n)$ be any 2 inputs into H, and let $y_1(n), y_2(n)$ be the corresponding outputs from system F, and let $r_1(n), r_2(n)$ be the corresponding outputs from system G.

Define $\hat{x}(n) \triangleq c_1 x_1(n) + c_2 x_2(n)$ for any $c_1, c_2 \in \mathbb{C}$.
Let $\hat{y}(n)$ be the corresponding output from F, and let $\hat{r}(n)$ be the corresponding output from G.

$\hat{y}(n) = c_1 y_1(n) + c_2 y_2(n)$ (by linearity of F)

so $\underline{\hat{r}(n) = c_1 r_1(n) + c_2 r_2(n)}$ (by linearity of G)

$\hookrightarrow \Rightarrow$ proves linearity of H

Define $\tilde{x}(n) \triangleq x_1(n-N)$ for $N \in \mathbb{Z}$. Let $\tilde{y}(n)$ and $\tilde{r}(n)$ be the corresponding outputs from F and G, respectively.

$\tilde{y}(n) = y_1(n-N)$ (by time-invariance of F)

So $\underline{\tilde{r}(n) = r_1(n-N)}$ (by time-invariance of G)

$\hookrightarrow \Rightarrow$ proves time-invariance of H

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(ii) (10 Points) Determine the impulse response values $h(n)$ and the frequency response values $H(\omega)$ of the filter H.

The frequency response of the cascade of LTI systems is the product of the individual frequency responses:

$$H(\omega) = F(\omega)G(\omega) = \left(\frac{1-\alpha}{1-\alpha e^{-i\omega}}\right)G(\omega) \quad \left(\text{using } F(\omega) \text{ from } 1.3(c)\right)$$

$$G(\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{-i\omega n}$$

$$= \sum_{n=-\infty}^{\infty}\left(\frac{1}{1-\alpha}\delta(n) - \frac{\alpha}{1-\alpha}\delta(n-1)\right)e^{-i\omega n}$$

$$= \frac{1}{1-\alpha}\left(\sum_{n=-\infty}^{\infty}\delta(n)e^{-i\omega n}\right) - \frac{\alpha}{1-\alpha}\left(\sum_{n=-\infty}^{\infty}\delta(n-1)e^{-i\omega n}\right)$$

$$= \frac{1}{1-\alpha}(1) - \frac{\alpha}{1-\alpha}\left(e^{-i\omega}\right) = \frac{1-\alpha e^{-i\omega}}{1-\alpha}.$$

$$\Rightarrow H(\omega) = \left(\frac{1-\alpha}{1-\alpha e^{-i\omega}}\right)\left(\frac{1-\alpha e^{-i\omega}}{1-\alpha}\right)$$

$$\boxed{H(\omega) = 1}$$

We know that $H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}$

$$\Rightarrow H(\omega) = \dots + h(-1)e^{i\omega} + h(0) + h(1)e^{-i\omega} + h(2)e^{-i\omega 2} + \dots$$

Since $H(\omega) = 1$, then by inspection, we must have:

$$h(n) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}, \quad \text{or} \quad \boxed{h(n) = \delta(n)}$$