EE 127 / EE 227AT

Solution Midterm 2 Spring 2016

Please write your answers on these sheets, use the back sides if needed. Show your work. You can use a fact from the slides/book without having to prove it unless you are specifically asked to do so. Be organized and use readable handwriting. There is a page for scratch work at the end.

Exercise 1 (Duality.) Consider the original problem $\min_{x\in\mathbb{R}^n} f_0(x)$ subject to $f_i(x) \leq 0$ for all $i = 1, ..., m$.

(a) (5 pts.) Write the Lagrangian function.

$L(x, \lambda) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x)$.

(b) (5 pts.) Write the dual function.

$g(\lambda) = \inf_x L(x, \lambda)$.

(c) (10 pts.) Prove that the dual function is concave.

We show that $-g(\lambda) = \sup_x -f_0(x) - \sum_{i=1}^m \lambda_i f_i(x)$ is convex. For each x, the term inside the maximization is affine in λ and thus also convex. Since the maximum of a collection of convex functions is a convex function, the conclusion follows.

(d) (10 pts.) Prove that for $\lambda \geq 0$, $g(\lambda)$ is no larger than the optimal value p^* of the original problem. You can assume that an optimal solution exists for the original problem.

If x is feasible in the original problem, then $f_i(x) \leq 0$ for all i. Thus, for such x,

$$g(\lambda) \leq f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \leq f_0(x).$$

Since this holds even for an optimal solution, the right-hand side can be made equal to p^* and the conclusion follows.

(e) (5 pts.) Suppose that $f_1(x) = \|x-a\|_2^2 \leq 0$. In this case, does Slater condition hold for the original problem? Explain.

In this case, only $x = a$ is feasible. But then $f_1(x) = 0$ and not negative as required by Slater. So the answer is no.

(f) (10 pts.) Suppose instead that $f_1(x) = \|x-a\|_2^2 - b \leq 0$ for $b > 0$ and that there are no other constraints. Also, let $f_0(x) = c^\top x$, for a nonzero vector c. Show that $x = -(\sqrt{b}/\|c\|_2)c + a$, with $\lambda = \|c\|_2/(2\sqrt{b})$, satisfies the KKT conditions.

The gradient of the Lagrangian gives $c + 2\lambda(x-a) = 0$. Thus, λ is not zero, which implies through complementary slackness that $\|x-a\|_2^2 = b$. Plugging in the given x, we see that this is indeed satisfied. The suggested λ is nonnegative so dual feasibility is satisfied. We only need to check the gradient condition:

$$c + 2\lambda(x-a) = c + 2\frac{\|c\|_2}{2\sqrt{b}}\left(-\frac{\sqrt{b}}{\|c\|_2}c + a - a\right) = 0.$$

(g) (10 pts.) Suppose that there are two candidate vectors for c. One with a small Euclidean length (case A) and one with a large Euclidean length (case B). Which case (A or B) will have an optimal value that is more sensitive to changes in the right-hand side of the constraint? Give an argument based on a quantitative estimate.

Sensitivity analysis tells us that $-\lambda$ is an estimate of how much the optimal value will change under changes in the right-hand side. Since λ is directly proportional to $\|c\|_2$ from above, case A will be less sensitive than case B.

Exercise 2 (Risk.) (10 pts.) In optimization problems involving superquantile risk measures, we have functions of the form $f(x) = x_n + (1/(1-\alpha))\sum_{j=1}^N p_j \max\{0, g(x, v^{(j)}) - x_n\}$, where $p_j \geq 0$, $\alpha \in [0,1)$, and x_n is the last component of x. Prove that if $g(x, v^{(j)})$ is convex in x for all $j = 1, ..., N$, then f is convex.

Since $\max\{0, z\}$ is an increasing convex function in z, the term $\max\{0, g(x, v^{(j)}) - x_n\}$ is a composition of an increasing function, with a convex function, which is convex. These terms are added together with nonnegative weights, which preserves convexity.

Exercise 3 (Nondifferentiable functions.) (10 pts.) Consider $f(x) = \max\{-x, x^2\}$. Give an explicit expression for the subdifferential of f at $x = 0$. Use an optimality condition to establish that $x = 0$ is optimal for f.

The subdifferential is $[-1, 0]$. Since the function is convex, a point is optimal if and only if zero is in the subdifferential at the point. This is obviously the case here.

Exercise 4 (Convex set.) (10 pts.) Show that the following set is a convex set:

$$\{x \in \mathbb{R}^n \ : \ \|x - a^{(i)}\|_2 \leq c^\top x + b^{(i)} \text{ for all } i = 1, ..., m\}.$$

The function $f_i(x) = \|x - a^{(i)}\|_2 - c^\top x - b^{(i)}$ is convex because every norm is convex and here it is simply composed with an affine function, which preserves convexity. The subtraction of an affine function further preserves convexity. Since sublevel sets of convex functions are convex sets, $\{x : f_i(x) \leq 0\}$ is a convex set. The conclusion then follows from realizing that the set in question is simply an intersection of such sets, which then must be convex.

Exercise 5 (Local optimality.) Give an example of an optimization problem on $\mathbb{R}$ with a locally optimal solution that is not a globally optimal solution in the following two cases. Give no picture. Write explicit formula.

1. (5 pts.) The objective function is convex.

   For example objective is $f(x) = x$ and the feasible set is $[-1, 0] \cup [1, 2]$. The point $x = 1$ is locally optimal, but not globally optimal.

2. (10 pts.) The feasible set is convex.

   For example objective is $f(x) = x$ everywhere except for $x = 0$ where $f(x) = -1$ and the feasible set is the whole real line. The point $x = 0$ is locally optimal, but not globally optimal.