EE 127 / EE 227AT

Midterm 2 Spring 2016

Please write your answers on these sheets, use the back sides if needed. Show your work. You can use a fact from the slides/book without having to prove it unless you are specifically asked to do so. Be organized and use readable handwriting. There is a page for scratch work at the end.

Exercise 1 (Duality.) Consider the original problem $\min_{x\in\mathbb{R}^n} f_0(x)$ subject to $f_i(x) \leq 0$ for all $i = 1, ..., m$.

(a) (5 pts.) Write the Lagrangian function.

(b) (5 pts.) Write the dual function.

(c) (10 pts.) Prove that the dual function is concave.

(d) (10 pts.) Prove that for $\lambda \geq 0$, $g(\lambda)$ is no larger than the optimal value p^* of the original problem. You can assume that an optimal solution exists for the original problem.

(e) (5 pts.) Suppose that $f_1(x) = \|x - a\|_2^2 \leq 0$. In this case, does Slater condition hold for the original problem? Explain.

(f) (10 pts.) Suppose instead that $f_1(x) = \|x - a\|_2^2 - b \leq 0$ for $b > 0$ and that there are no other constraints. Also, let $f_0(x) = c^\top x$, for a nonzero vector c. Show that $x = -(\sqrt{b}/\|c\|_2)c + a$, with $\lambda = \|c\|_2/(2\sqrt{b})$, satisfies the KKT conditions.

(g) (10 pts.) Suppose that there are two candidate vectors for c. One with a small Euclidean length (case A) and one with a large Euclidean length (case B). Which case (A or B) will have an optimal value that is more sensitive to changes in the right-hand side of the constraint? Give an argument based on a quantitative estimate.

Exercise 2 (Risk.) (10 pts.) In optimization problems involving superquantile risk measures, we have functions of the form $f(x) = x_n + (1/(1-\alpha)) \sum_{j=1}^{N} p_j \max\{0, g(x, v^{(j)}) - x_n\}$, where $p_j \geq 0$, $\alpha \in [0, 1)$, and x_n is the last component of x. Prove that if $g(x, v^{(j)})$ is convex in x for all $j = 1, ..., N$, then f is convex.

Exercise 3 (Nondifferentiable functions.) (10 pts.) Consider $f(x) = \max\{-x, x^2\}$. Give an explicit expression for the subdifferential of f at $x = 0$. Use an optimality condition to establish that $x = 0$ is optimal for f.

Exercise 4 (Convex set.) (10 pts.) Show that the following set is a convex set:

$$\{x \in \mathbb{R}^n \ : \ \|x - a^{(i)}\|_2 \leq c^\top x + b^{(i)} \text{ for all } i = 1, ..., m\}.$$

Exercise 5 (Local optimality.) Give an example of an optimization problem on $\mathbb{R}$ with a locally optimal solution that is not a globally optimal solution in the following two cases. Give no picture. Write explicit formula.

1. (5 pts.) The objective function is convex.

2. (10 pts.) The feasible set is convex.

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